JMO 2012/5

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Twitch Solves ISL

Episode 36

Problem

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

Video

https://youtu.be/5cIgTuYBWKo

External Link

https://aops.com/community/p2669967

Solution

The answer is $S = 502$ (not 503!).

Claim. If $\gcd(k, 2012) = 1$, then necessarily either k or $2012 - k$ will counts towards S.

Proof. First note that both ak, bk are nonzero modulo 2012. Note also that $ak \not\equiv bk \pmod{2012}$.

So if r_a is the remainder of $ak \pmod{2012}$, then $2012 - r_a$ is the remainder of $a(2012-k) \pmod{2012}$ Similarly we can consider r_b and $2012 - r_b$. As mentioned already, we have $r_a \neq r_b$. So either $r_a > r_b$ or $2012 - r_a > 2012 - r_b$. □

This implies $S \geq \frac{1}{2}\varphi(2012) = 502$.

But this can actually be achieved by taking $a = 4$ and $b = 1010$, since

- If k is even, then $ak \equiv bk \pmod{2012}$ so no even k counts towards S; and
- If $k \equiv 0 \pmod{503}$, then $ak \equiv 0 \pmod{2012}$ so no such k counts towards S.

This gives the final answer $S \geq 502$.

Remark. A similar proof works with 2012 replaced by any n and will give an answer of $\frac{1}{2}\varphi(n)$. For composite n, one uses the Chinese remainder theorem to pick distinct a and b not divisible by n such that $\operatorname{lcm}(a - b, a) = n$.