COMC 2020/C4

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Twitch Solves ISL

Episode 36

Problem

Let $S = \{4, 8, 9, 16, \dots\}$ be the set of integers of the form m^k for integers $m, k \geq 2$. For a positive integer n, let $f(n)$ denote the number of ways to write n as the sum of (one or more) distinct elements of S.

(a) Prove that $f(30) = 0$.

(b) Show that $f(n) \geq 1$ for $n \geq 31$.

(c) Let T be the set of integers n for which $f(n) = 3$. Prove that T is finite and nonempty, and find $\max(T)$.

Video

https://youtu.be/8g21cEWiomg

Solution

We define S' as S with powers of 2 removed, so that

$$\begin{aligned} S' &= \{9, 25, 36, 49, 81, 100, 121, 144, 169, 196, 225\} \\ &\cup \{27, 125, 216, 343\} \\ &\cup \{243, \dots\} \end{aligned}$$

The main idea is that every multiple of 4 has a unique representation as a sum of powers of 2 larger than 2, while no other number can be represented at all. In other words:

Claim. We have $f(n)$ is the number of ways to write n as the sum of distinct elements of S, plus some multiple of 4.

Part (a) is clear now.
For part (b), note that:

- for $n \equiv 0 \pmod 4$, we vacuously have $f(n) > 0$;
- for $n \equiv 1 \pmod 4$, we have $n - 9 \equiv 0 \pmod 4$ so $f(n) > 0$ for $n \geq 9$;
- for $n \equiv 2 \pmod 4$, we have $n - (9 + 25) \equiv 0 \pmod 4$ so $f(n) > 0$ for $n \geq 34$;
- for $n \equiv 3 \pmod 4$, we have $n - 27 \equiv 0 \pmod 4$ so $f(n) > 0$ for $n \geq 27$.

So this finishes part (b).
The proof of part (c) is similar but more annoying.

- For $n \equiv 0 \bmod 4$: all of n, $n - (9 + 27)$, $n - 36$, $n - (25 + 27)$ are divisible by 4, so $f(n) > 3$ for $n \geq 52$.
- For $n \equiv 1 \bmod 4$: all of $n - 9$, $n - 25$, $n - (9 + 36)$, $n - 49$ are divisible by 4; so $f(n) > 3$ for $n \geq 49$.
- For $n \equiv 2 \bmod 4$: all of $n - (9 + 25)$, $n - (9 + 49)$, $n - (9 + 25 + 36)$, $n - (25 + 49)$ are divisible by 4; so $f(n) > 3$ for $n \geq 74$.
- For $n \equiv 3 \bmod 4$: all of $n - 27$, $n - (27 + 36)$, $n - (9 + 25 + 49)$, $n - (9 + 25 + 81)$ are divisible by 4; so $f(n) > 3$ for $n \geq 115$.

On the other hand, one can check manually that $f(111) = 3$, by exhaustively checking that there are no smaller combinations of elements of S' that add up to a $3 \pmod 4$ number less than 111 (i.e. that $9 + 25 + 81 = 115$ really is the fourth smallest possible). So $f(111) = 3$ follows and part (c) is completed.