# COMC 2020/C4 

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## Twitch Solves ISL

Episode 36

## Problem

Let $S=\{4,8,9,16, \ldots\}$ be the set of integers of the form $m^{k}$ for integers $m, k \geq 2$. For a positive integer $n$, let $f(n)$ denote the number of ways to write $n$ as the sum of (one or more) distinct elements of $S$.
(a) Prove that $f(30)=0$.
(b) Show that $f(n) \geq 1$ for $n \geq 31$.
(c) Let $T$ be the set of integers $n$ for which $f(n)=3$. Prove that $T$ is finite and nonempty, and find $\max (T)$.

## Video

https://youtu.be/8g21cEWiomg

## Solution

We define $S^{\prime}$ as $S$ with powers of 2 removed, so that

$$
\begin{aligned}
S^{\prime} & =\{9,25,36,49,81,100,121,144,169,196,225\} \\
& \cup\{27,125,216,343\} \\
& \cup\{243, \ldots\}
\end{aligned}
$$

The main idea is that every multiple of 4 has a unique representation as a sum of powers of 2 larger than 2 , while no other number can be represented at all. In other words:

Claim. We have $f(n)$ is the number of ways to write $n$ as the sum of distinct elements of $S$, plus some multiple of 4 .

Part (a) is clear now.
For part (b), note that:

- for $n \equiv 0(\bmod 4)$, we vacuously have $f(n)>0$;
- for $n \equiv 1(\bmod 4)$, we have $n-9 \equiv 0(\bmod 4)$ so $f(n)>0$ for $n \geq 9$;
- for $n \equiv 2(\bmod 4)$, we have $n-(9+25) \equiv 0(\bmod 4)$ so $f(n)>0$ for $n \geq 34$;
- for $n \equiv 3(\bmod 4)$, we have $n-27 \equiv 0(\bmod 4)$ so $f(n)>0$ for $n \geq 27$.

So this finishes part (b).
The proof of part (c) is similar but more annoying.

- For $n \equiv 0 \bmod 4:$ all of $n, n-(9+27), n-36, n-(25+27)$ are divisible by 4 , so $f(n)>3$ for $n \geq 52$.
- For $n \equiv 1 \bmod 4:$ all of $n-9, n-25, n-(9+36), n-49$ are divisible by 4 ; so $f(n)>3$ for $n \geq 49$.
- For $n \equiv 2 \bmod 4:$ all of $n-(9+25), n-(9+49), n-(9+25+36), n-(25+49)$ are divisible by 4 ; so $f(n)>3$ for $n \geq 74$.
- For $n \equiv 3 \bmod 4:$ all of $n-27, n-(27+36), n-(9+25+49), n-(9+25+81)$ are divisible by 4 ; so $f(n)>3$ for $n \geq 115$.

On the other hand, one can check manually that $f(111)=3$, by exhaustively checking that there are no smaller combinations of elements of $S^{\prime}$ that add up to a $3(\bmod 4)$ number less than 111 (i.e. that $9+25+81=115$ really is the fourth smallest possible). So $f(111)=3$ follows and part (c) is completed.

