USEMO 2020/5

Evan Chen

Twitch Solves ISL

Episode 35

Problem

The sides of a convex 200-gon $A_1A_2 \dots A_{200}$ are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

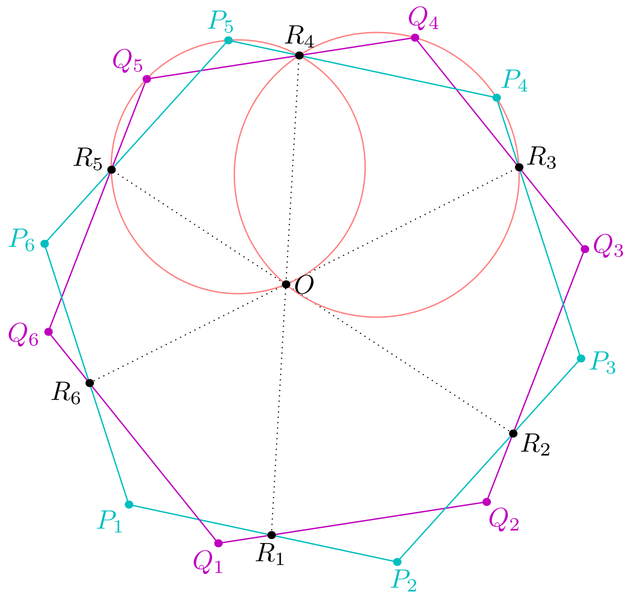
Prove that the 50 diagonals $\overline{A_1A_{101}}$, $\overline{A_3A_{103}}$, $\dots$, $\overline{A_{99}A_{199}}$ are concurrent.

Video

https://youtu.be/uj93tNL8f7M

Solution

We present a diagram (with 100 replaced by 6, for simplicity).



Let $P_1 \cdots P_{100}$ and $Q_1 \cdots Q_{100}$ be the regular 100-gons (oriented counterclockwise), and define $R_i = \overline{P_iP_{i+1}} \cap \overline{Q_iQ_{i+1}}$ for all i, where all indices are taken modulo 100. We wish to show that $\overline{R_1R_{51}}, \ldots, \overline{R_{50}R_{100}}$ are concurrent.

Let O be the spiral center taking $P_1 \cdots P_{100} \to Q_1 \cdots Q_{100}$ (it exists since the 100-gons are not homothetic). We claim that O is the desired concurrency point.

Claim. $\measuredangle R_iOR_{i+1} = \frac{\pi}{50}$ for all i.

Proof. Since $\triangle OP_iP_{i+1} \overset{+}{\sim} \triangle OQ_iQ_{i+1}$, we have $\triangle OP_iQ_i \overset{+}{\sim} \triangle OP_{i+1}Q_{i+1}$, so O, R_i, P_{i+1}, Q_{i+1} are concyclic. Similarly O, R_{i+1}, P_{i+1}, Q_{i+1} are concyclic, so

$$\measuredangle R_iOR_{i+1} = \measuredangle R_iP_{i+1}R_{i+1} = \frac{\pi}{50}$$

as wanted. $\square$

It immediately follows that O lies on all 50 diagonals $\overline{R_iR_{i+50}}$, as desired.