USEMO 2020/5

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Twitch Solves ISL

Episode 35

Problem

The sides of a convex 200-gon $A_1A_2\dots A_{200}$ are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

Prove that the 50 diagonals $\overline{A_1A_{101}}$, $\overline{A_3A_{103}}$, $\dots$, $\overline{A_{99}A_{199}}$ are concurrent.

Video

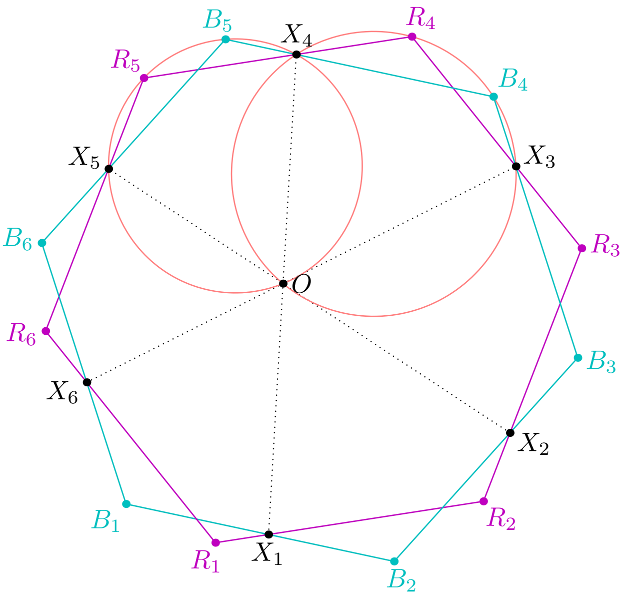
https://youtu.be/5a_XCGKiXnI

External Link

https://aops.com/community/p18486857

Solution

We present a diagram (with 100 replaced by 6, for simplicity).



Let $B_1 \dots B_{100}$ and $R_1 \dots R_{100}$ be the regular 100-gons (oriented counterclockwise), and define $X_i = A_{2i+1} = \overline{B_iB_{i+1}} \cap \overline{R_iR_{i+1}}$ for all i, where all indices are taken modulo 100. We wish to show that $\overline{X_1X_{51}}$, $\dots$, $\overline{X_{50}X_{100}}$ are concurrent.

We now present two approaches.

First approach (by spiral similarity). Let O be the spiral center taking $B_1 \dots B_{100} \to R_1 \dots R_{100}$ (it exists since the 100-gons are not homothetic). We claim that O is the desired concurrency point.

Claim. $\measuredangle X_iOX_{i+1} = \frac{\pi}{50}$ for all i.

Proof. Since $\triangle OB_iB_{i+1} \overset{+}{\sim} \triangle OR_iR_{i+1}$, we have $\triangle OB_iR_i \overset{+}{\sim} \triangle OB_{i+1}R_{i+1}$, so O, X_i, B_{i+1}, R_{i+1} are concyclic. Similarly O, X_{i+1}, B_{i+1}, R_{i+1} are concyclic, so

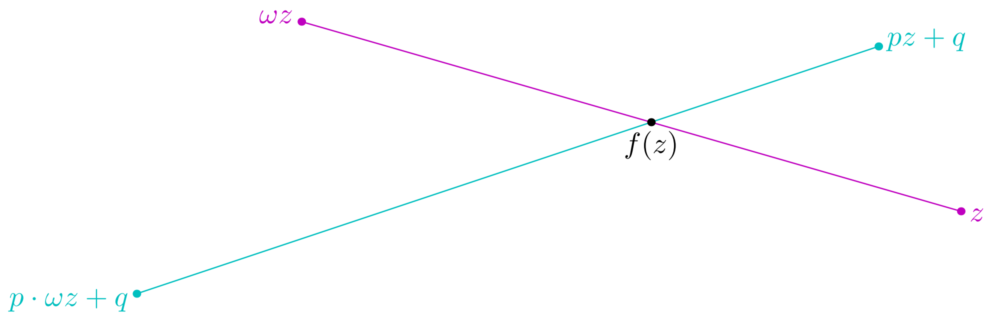
$$\measuredangle X_iOX_{i+1} = \measuredangle X_iB_{i+1}X_{i+1} = \frac{\pi}{50}$$

as wanted. $\square$

It immediately follows that O lies on all 50 diagonals $\overline{X_iX_{i+50}}$, as desired.

Second approach (by complex numbers). Let ω be a primitive 100th root of unity. We will impose complex coordinates so that $R_k = \omega^k$, while $B_k = p\omega^k + q$, where m and b are given constant complex numbers.

In general for $|z| = 1$, we will define $f(z)$ as the intersection of the line through z and ωz, and the line through $pz + q$ and $p \cdot \omega z + q$.



In particular, X_k is $f(\omega^k)$.

Claim. There exist complex numbers a, b, c such that $f(z) = a + bz + cz^2$, for every $|z| = 1$.

Proof. Since $f(z)$ and $\frac{f(z)-q}{p}$ both lie on the chord joining z to ωz we have

$$\begin{aligned} z + \omega z &= f(z) + \omega z^2 \cdot \overline{f(z)} \\ z + \omega z &= \frac{f(z) - q}{p} + \omega z^2 \cdot \frac{\overline{f(z)} - \overline{q}}{\overline{p}}. \end{aligned}$$

Subtracting the first equation from the $\overline{p}$ times the second to eliminate $\overline{f(z)}$, we get that $f(z)$ should be a degree-two polynomial in z (where p and q are fixed constants). $\square$

Claim. Let $f(z) = a + bz + cz^2$ as before. Then the locus of lines through $f(z)$ and $f(-z)$, as $|z| = 1$ varies, passes through a fixed point.

Proof. By shifting we may assume $a = 0$, and by scaling we may assume b is real (i.e. $\overline{b} = b$). Then the point $-\overline{c}$ works, since

$$\frac{f(z) + \overline{c}}{f(-z) + \overline{c}} = \frac{\overline{c} + bz + cz^2}{\overline{c} - bz + cz^2}$$

is real — it obviously equals its own conjugate. (Alternatively, without the assumptions $a = 0$ and $b \in \mathbb{R}$, the fixed point is $a - \frac{b\overline{c}}{\overline{b}}$.) $\square$

Remark (We know a priori the exact coefficients shouldn't matter)**.** In fact, the exact value is

$$f(z) = \frac{-\omega\overline{q}z^2 + (1 - \overline{p})(1 + \omega)z - \frac{\overline{p}}{p}q}{1 - \frac{\overline{p}}{p}}.$$

Since p and q could be any complex numbers, the quantity c/b (which is all that matters for concurrence) could be made to be equal to any value. For this reason, we know *a priori* the exact coefficients should be irrelevant.