USEMO 2020/4

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Twitch Solves ISL

Episode 35

Problem

A function f from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

Video

https://youtu.be/uj93tNL8f7M

Solution

We present two solutions.

First solution (Nikolai Beluhov) We first begin with the following observation.

Claim. We must have $f(y) \ge y$ for all $y > 0$.

Proof. Otherwise, choose $0 < x < 1$ satisfying that $f(y) = (1-x) \cdot y$. Then plugging this $P(x,y)$ gives $xf(y) = 0$, contradiction. $\square$

Now, we make the substitution $f(x) = x + g(x)$, so that g is a function $\mathbb{R}_{>0} \to \mathbb{R}_{\ge 0}$. The given function equation reads $g(x+xy+(y+g(y))) + x + (y+g(y)) = (xy+xg(y)) + (x+y+g(x+y))$, or

$$g(x+y+xy+g(y)) = (x-1)g(y) + g(x+y). \qquad (\dagger)$$

We have to show that g is the zero function from $(\dagger)$.

Claim (Injectivity for nonzero outputs)**.** If $g(a) = g(b)$ for $a \ne b$, then we must actually have $g(a) = g(b) = 0$.

Proof. Setting (a,b) and (b,a) in $(\dagger)$ gives $(a-1)g(b) = (b-1)g(a)$ which, since $a-1 \ne b-1$, forces $g(a) = g(b) = 0$. $\square$

Claim (g vanishes on $(1,\infty)$)**.** We have $g(t) = 0$ for $t > 1$.

Proof. If we set $x = 1$ in $(\dagger)$ we obtain that $g(g(y)+2y+1) = g(1+y)$. As the inputs are obviously unequal, the previous claim gives $g(1+y) = 0$ for all $y > 0$. $\square$

Now $x = 2$ in $(\dagger)$ to get $g(y) = 0$, as needed.

Second solution (from authors) We start with the same opening of showing $f(y) \ge y$, defining $f(x) = x + g(x)$, so g satisfies $(\dagger)$. Here is another proof that $g \equiv 0$ from $(\dagger)$.

Claim. If g is not the zero function, then for any constant C, we have $g(t) > C$ for sufficiently large t.

Proof. In $(\dagger)$ fix y to be any input for which $g(y) > 0$. Then

$$g\left((1+y)x + (y+g(y))\right) \ge (x-1)g(y)$$

so for large x, we get the conclusion. $\square$

Remark. You could phrase the lemma succinctly as "$\lim_{x\to\infty} g(x) = +\infty$". But I personally think it's a bit confusing to do so because in practice we usually talk about limits of continuous (or well-behaved) functions, so a statement like this would have the wrong connotations, even if technically correct and shorter.

On the other hand, by choosing $x = 1$ and $y = t-1$ for $t > 1$ in $(\dagger)$, we get

$$g(2z + g(z) - 2) = g(z)$$

and hence one can generate an infinite sequence of fixed points: start from $z_0 = 100$, and define $z_n = 2z_{n-1} + g(z_{n-1}) - 2 > z_{n-1} + 98$ for $n \ge 1$ to get

$$g(z_0) = g(z_1) = g(z_2) = \cdots$$

and since the z_i are arbitrarily large, this produces a contradiction if $g \not\equiv 0$.