# USEMO 2020/4 

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## Twitch Solves ISL

Episode 35

## Problem

A function $f$ from the set of positive real numbers to itself satisfies

$$
f(x+f(y)+x y)=x f(y)+f(x+y)
$$

for all positive real numbers $x$ and $y$. Prove that $f(x)=x$ for all positive real numbers $x$.

## Video

https://youtu.be/5a_XCGKiXnI

## External Link

https://aops.com/community/p18486884

## Solution

We present two solutions.

First solution (Nikolai Beluhov). We first begin with the following observation.
Claim. We must have $f(y) \geq y$ for all $y>0$.
Proof. Otherwise, choose $0<x<1$ satisfying that $f(y)=(1-x) \cdot y$. Then plugging this $P(x, y)$ gives $x f(y)=0$, contradiction.

Now, we make the substitution $f(x)=x+g(x)$, so that $g$ is a function $\mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$. The given function equation reads $g(x+x y+(y+g(y)))+x+(y+g(y))=(x y+x g(y))+$ $(x+y+g(x+y))$, or

$$
g(x+y+x y+g(y))=(x-1) g(y)+g(x+y) .
$$

We have to show that $g$ is the zero function from ( $\dagger$ ).
Claim (Injectivity for nonzero outputs). If $g(a)=g(b)$ for $a \neq b$, then we must actually have $g(a)=g(b)=0$.

Proof. Setting $(a, b)$ and $(b, a)$ in $(\dagger)$ gives $(a-1) g(b)=(b-1) g(a)$ which, since $a-1 \neq$ $b-1$, forces $g(a)=g(b)=0$.

Claim ( $g$ vanishes on $(1, \infty)$ ). We have $g(t)=0$ for $t>1$.
Proof. If we set $x=1$ in $(\dagger)$ we obtain that $g(g(y)+2 y+1)=g(1+y)$. As the inputs are obviously unequal, the previous claim gives $g(1+y)=0$ for all $y>0$.

Now $x=2$ in ( $\dagger$ ) to get $g(y)=0$, as needed.
Second solution (from authors). We start with the same opening of showing $f(y) \geq y$, defining $f(x)=x+g(x)$, so $g$ satisfies $(\dagger)$. Here is another proof that $g \equiv 0$ from ( $\dagger$ ).
Claim. If $g$ is not the zero function, then for any constant $C$, we have $g(t)>C$ for sufficiently large $t$.

Proof. In ( $\dagger$ ) fix $y$ to be any input for which $g(y)>0$. Then

$$
g((1+y) x+(y+g(y))) \geq(x-1) g(y)
$$

so for large $x$, we get the conclusion.
Remark. You could phrase the lemma succinctly as " $\lim _{x \rightarrow \infty} g(x)=+\infty$ ". But I personally think it's a bit confusing to do so because in practice we usually talk about limits of continuous (or well-behaved) functions, so a statement like this would have the wrong connotations, even if technically correct and shorter.

On the other hand, by choosing $x=1$ and $y=t-1$ for $t>1$ in $(\dagger)$, we get

$$
g(2 t+g(z)-1)=g(t)
$$

and hence one can generate an infinite sequence of fixed points: start from $t_{0}=100$, and define $t_{n}=2 t_{n-1}+g\left(t_{n-1}\right)-2>t_{n-1}+98$ for $n \geq 1$ to get

$$
g\left(t_{0}\right)=g\left(t_{1}\right)=g\left(t_{2}\right)=\cdots
$$

and since the $t_{i}$ are arbitrarily large, this produces a contradiction if $g \not \equiv 0$.

