# USEMO 2020/4

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Episode 35

## Problem

A function $f$ from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers $x$ and $y$. Prove that $f(x) = x$ for all positive real numbers $x$.

## Video

https://youtu.be/5a_XCGKiXnI

## External Link

https://aops.com/community/p18486884

## Solution

We present two solutions.

**First solution (Nikolai Beluhov).** We first begin with the following observation.

**Claim.** We must have $f(y) \geq y$ for all $y > 0$.

*Proof.* Otherwise, choose $0 < x < 1$ satisfying that $f(y) = (1-x) \cdot y$. Then plugging this $P(x, y)$ gives $xf(y) = 0$, contradiction. $\square$

Now, we make the substitution $f(x) = x + g(x)$, so that $g$ is a function $\mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$. The given function equation reads $g(x+xy+(y+g(y)))+x+(y+g(y)) = (xy+xg(y))+(x+y+g(x+y))$, or

$$g(x+y+xy+g(y)) = (x-1)g(y) + g(x+y). \qquad (\dagger)$$

We have to show that $g$ is the zero function from ($\dagger$).

**Claim** (Injectivity for nonzero outputs)**.** If $g(a) = g(b)$ for $a \neq b$, then we must actually have $g(a) = g(b) = 0$.

*Proof.* Setting $(a, b)$ and $(b, a)$ in ($\dagger$) gives $(a-1)g(b) = (b-1)g(a)$ which, since $a - 1 \neq b - 1$, forces $g(a) = g(b) = 0$. $\square$

**Claim** ($g$ vanishes on $(1, \infty)$)**.** We have $g(t) = 0$ for $t > 1$.

*Proof.* If we set $x = 1$ in ($\dagger$) we obtain that $g(g(y) + 2y + 1) = g(1 + y)$. As the inputs are obviously unequal, the previous claim gives $g(1 + y) = 0$ for all $y > 0$. $\square$

Now $x = 2$ in ($\dagger$) to get $g(y) = 0$, as needed.

**Second solution (from authors).** We start with the same opening of showing $f(y) \geq y$, defining $f(x) = x + g(x)$, so $g$ satisfies ($\dagger$). Here is another proof that $g \equiv 0$ from ($\dagger$).

**Claim.** If $g$ is not the zero function, then for any constant $C$, we have $g(t) > C$ for sufficiently large $t$.

*Proof.* In ($\dagger$) fix $y$ to be any input for which $g(y) > 0$. Then

$$g\left((1+y)x + (y + g(y))\right) \geq (x-1)g(y)$$

so for large $x$, we get the conclusion. $\square$

**Remark.** You could phrase the lemma succinctly as "$\lim_{x\to\infty} g(x) = +\infty$". But I personally think it's a bit confusing to do so because in practice we usually talk about limits of continuous (or well-behaved) functions, so a statement like this would have the wrong connotations, even if technically correct and shorter.

On the other hand, by choosing $x = 1$ and $y = t - 1$ for $t > 1$ in ($\dagger$), we get

$$g(2t + g(z) - 1) = g(t)$$

and hence one can generate an infinite sequence of fixed points: start from $t_0 = 100$, and define $t_n = 2t_{n-1} + g(t_{n-1}) - 2 > t_{n-1} + 98$ for $n \geq 1$ to get

$$g(t_0) = g(t_1) = g(t_2) = \cdots$$

and since the $t_i$ are arbitrarily large, this produces a contradiction if $g \not\equiv 0$.