USEMO 2020/3

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Twitch Solves ISL

Episode 34

Problem

Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

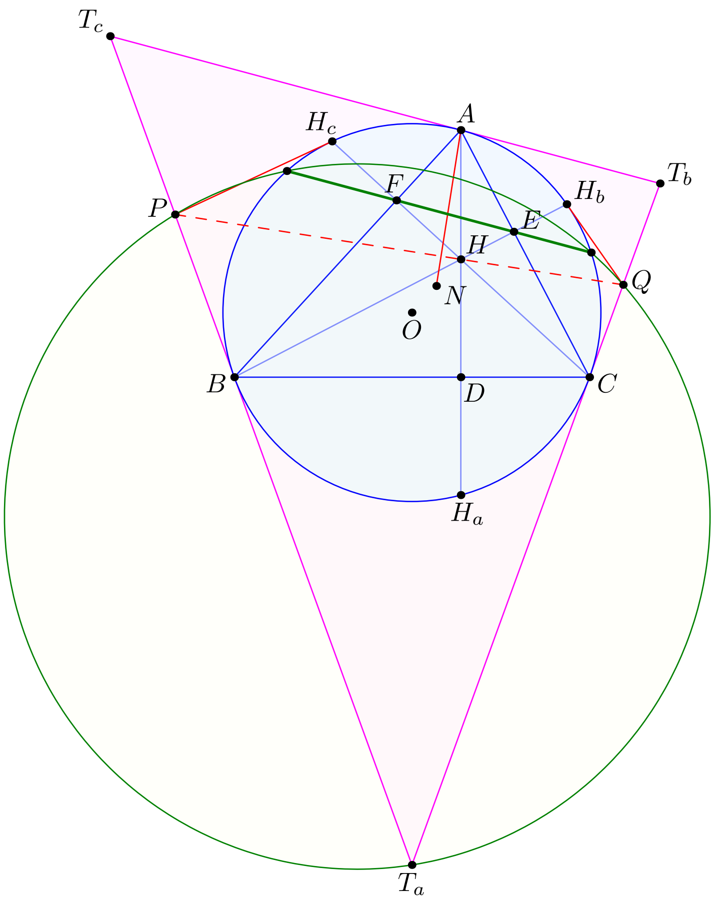
Video

https://youtu.be/uj93tNL8f7M

Solution

We begin by introducing several notations. The orthic triangle is denoted DEF and the tangential triangle is denoted $T_aT_bT_c$. The reflections of H across the sides are denoted H_a, H_b, H_c. We also define the crucial points P and Q as the poles of $\overline{H_cB}$ and $\overline{H_bC}$ with respect to Γ.

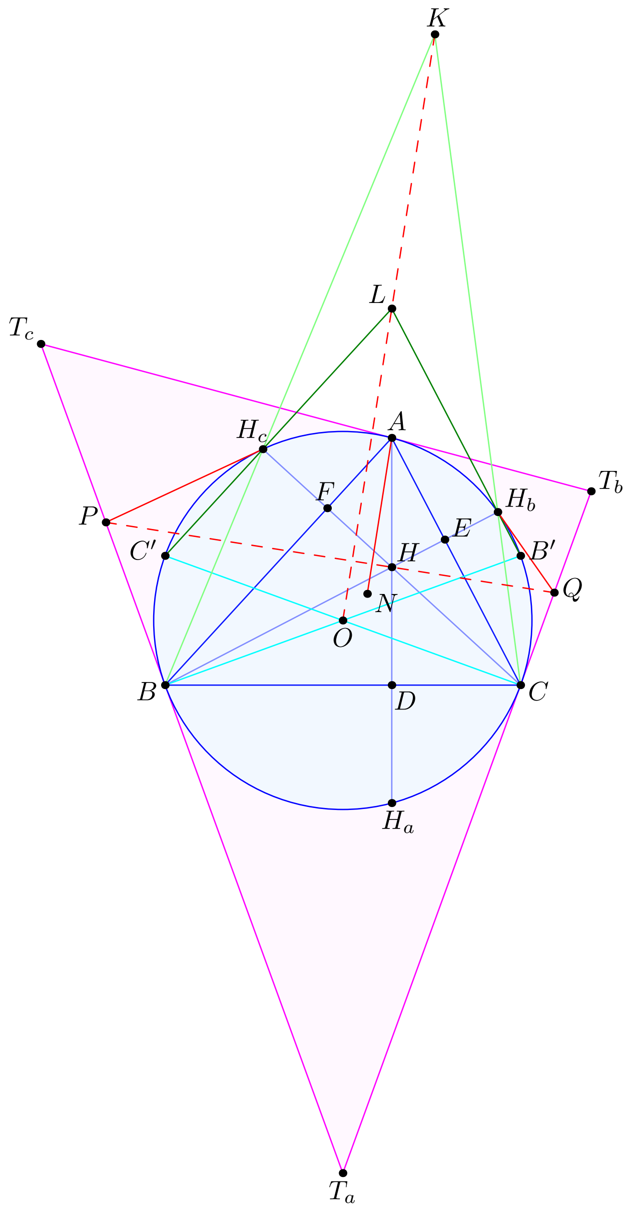
The solution, based on the independent solutions found by Anant Mudgal and Nikolai Beluhov, hinges on two central claims: that ω_A is the circumcircle of $\triangle T_aPQ$, and that $\overline{EF}$ is the radical axis of Γ and ω_A. We prove these two claims in turn.



Claim (Characterization of ω_A)**.** Line PQ passes through H and is perpendicular to $\overline{AN}$.

Proof. The fact that H lies on line PQ is immediate by Brokard's theorem.

Showing the perpendicularity is the main part. Denote by B' and C' the antipodes of B and C on Γ. Also, define $L = \overline{H_cC'} \cap \overline{H_bB'}$ and $K = \overline{BH_c} \cap \overline{CH_b}$, as shown.



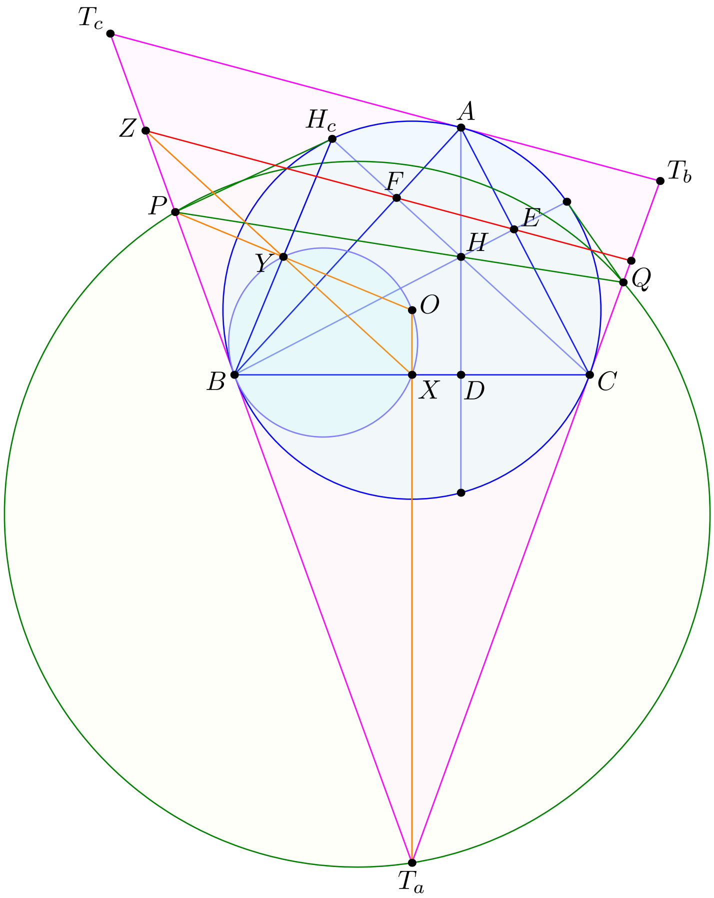
We observe that:

- We have $\overline{OK} \perp \overline{PQ}$ since K is the pole of line $\overline{PQ}$ (again by Brokard).
- The points O, K, L are collinear by Pascal's theorem on $BH_cC'CH_bB'$.
- The point L is seen to be the reflection of H across A, so it follows $\overline{AN} \parallel \overline{OL}$ by a $\frac{1}{2}$-factor homothety at A.

Putting these three observations together completes the solution. □

Claim (Radical axis of ω_A and Γ)**.** Line EF coincides with the radical axis of ω_A and Γ.

Proof. Let lines EF and T_aT_c meet at Z. It suffices to show Z lies on the radical axis, and then repeat the argument on the other side.



Since $\measuredangle FBZ = \measuredangle ABZ = \measuredangle BCA = \measuredangle EFA = \measuredangle ZFB$, it follows $ZB = ZF$. We introduce two other points X and Y on the perpendicular bisector of $\overline{BF}$: they are the midpoints of $\overline{BC}$ and $\overline{BH_c}$.

Since $OX \cdot OT_a = OB^2 = OY \cdot OP$, it follows that $XYPT_a$ is cyclic. Then

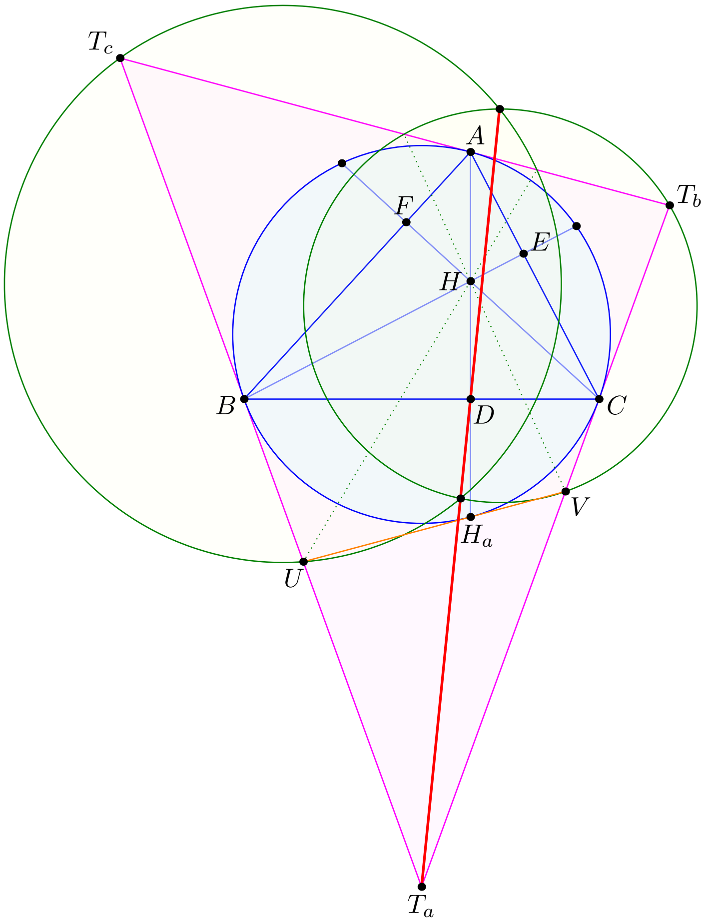
$$ZP \cdot ZT_a = ZX \cdot ZY = ZB^2$$

with the last equality since the circumcircle of $\triangle BXY$ is tangent to Γ (by a $\frac{1}{2}$-homothety at B). So the proof of the claim is complete. $\square$

Finally, we are ready to finish the problem.

Claim. Line DT_a coincides with the radical axis of ω_B and ω_C.

Proof. The point D already coincides with the radical axis because it is the radical center of Γ, ω_B and ω_C. As for the point T, we let the tangent to Γ at H_a meet $\overline{T_aT_c}$ at U and V; by the first claim, these lie on ω_C and ω_B respectively.



We need to show $T_aU \cdot T_aT_c = T_aV \cdot T_aT_b$.

But UVT_bT_c is apparently cyclic: the sides $\overline{T_bT_c}$ and $\overline{UV}$ are reflections across a line perpendicular to $\overline{AH_a}$, while the sides $\overline{UT_c}$ and $\overline{VT_b}$ are reflections across a line perpendicular to $\overline{BC}$. So this is true. $\square$

Now since $\triangle DEF$ and $\triangle XYZ$ are homothetic (their opposite sides are parallel), and their incenters are respectively H and O, the problem is solved.

Remark (Nikolai Beluhov — generalization with variable ABC and fixed H)**.** Take a fixed circle Γ and a fixed point H in its interior. Then there exist infinitely many triangles ABC with orthocenter H and circumcircle Γ. In fact, for every point A on Γ we get a unique pair of B and C, determined as follows: Let line AH meet Γ again at S_A, and take B and C to be the intersection points of the perpendicular bisector of segment HS_A with Γ.

With this framework, the following generalization is true: The radical center W of ω_A, ω_B, and ω_C is the same point for all such triangles. Indeed, the Euler circle e of triangle ABC is constant because it depends only on H and Γ. Let inversion relative to Γ map e onto Ω. Then all three of T_a, T_b, and T_c lie on Ω, and so, by the solution, W is the homothety center of e and Ω. (We take the homothety center with positive ratio when triangle ABC is acute, and with negative ratio when it is obtuse. When triangle ABC is right-angled, Ω degenerates into a straight line.)

Explicitly, let Γ be the unit circle and put H on the real axis at h. Then W is also on the real axis, at $4h/(h^2+3)$.

Furthermore, it turns out the power of W with respect to ω_A, ω_B, and ω_C is constant as well. This, however, is much tougher to prove; we are not aware of a purely geometric proof at this time. Explicitly, in the setting above where Γ is the unit circle and H is on the real axis at h, the power of W with respect to ω_A, ω_B, and ω_C equals $12(h^2-1)/(h^2+3)^2$.