USEMO 2020/3

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Twitch Solves ISL

Episode 34

Problem

Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

Video

https://youtu.be/5a_XCGKiXnI

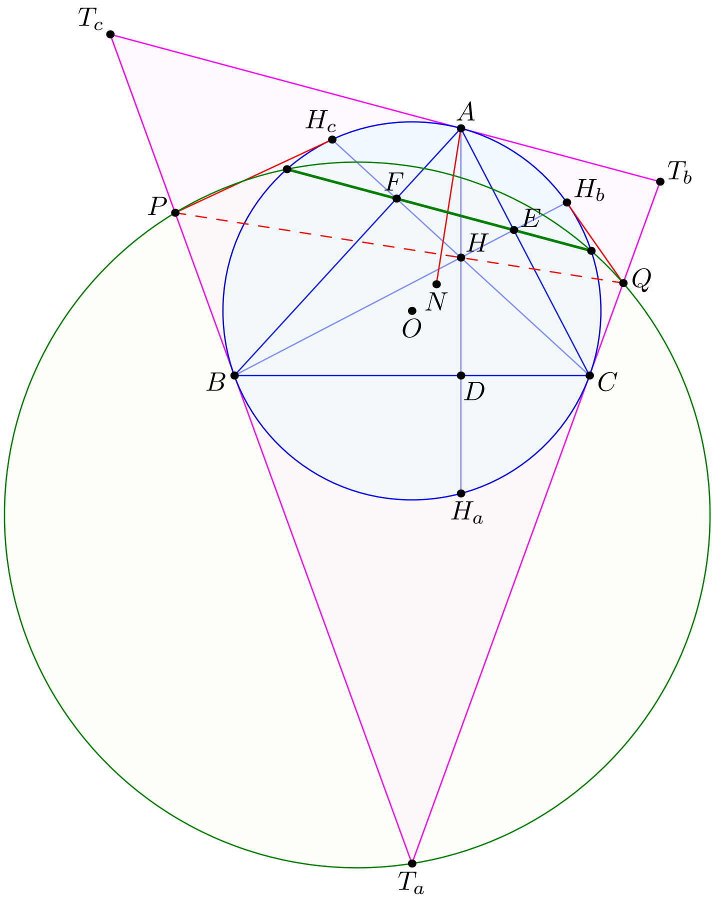
External Link

https://aops.com/community/p18471120

Solution

We begin by introducing several notations. The orthic triangle is denoted DEF and the tangential triangle is denoted $T_aT_bT_c$. The reflections of H across the sides are denoted H_a, H_b, H_c. We also define the crucial points P and Q as the poles of $\overline{H_cB}$ and $\overline{H_bC}$ with respect to Γ.

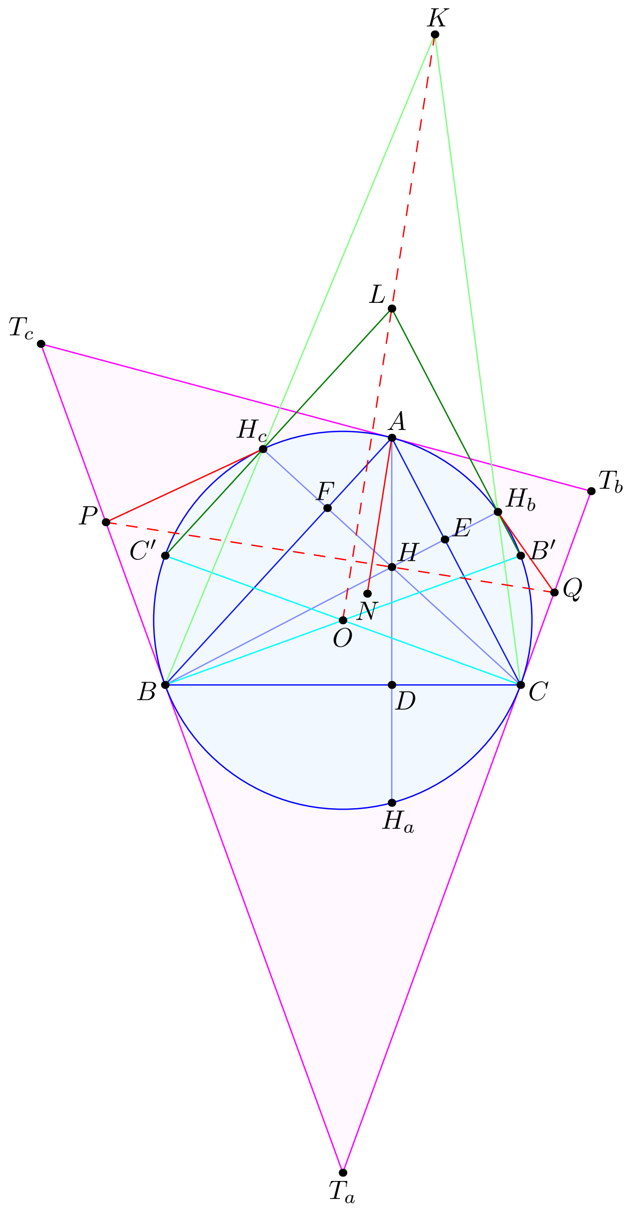
The solution, based on the independent solutions found by Anant Mudgal and Nikolai Beluhov, hinges on two central claims: that ω_A is the circumcircle of $\triangle T_aPQ$, and that $\overline{EF}$ is the radical axis of Γ and ω_A. We prove these two claims in turn.



Claim (Characterization of ω_A)**.** Line PQ passes through H and is perpendicular to $\overline{AN}$.

Proof. The fact that H lies on line PQ is immediate by Brokard's theorem.

Showing the perpendicularity is the main part. Denote by B' and C' the antipodes of B and C on Γ. Also, define $L = \overline{H_cC'} \cap \overline{H_bB'}$ and $K = \overline{BH_c} \cap \overline{CH_b}$, as shown.



We observe that:

- We have $\overline{OK} \perp \overline{PQ}$ since K is the pole of line $\overline{PQ}$ (again by Brokard).
- The points O, K, L are collinear by Pascal's theorem on $BH_cC'CH_bB'$.
- The point L is seen to be the reflection of H across A, so it follows $\overline{AN} \parallel \overline{OL}$ by a $\frac{1}{2}$-factor homothety at H.

Putting these three observations together completes the first claim. $\square$

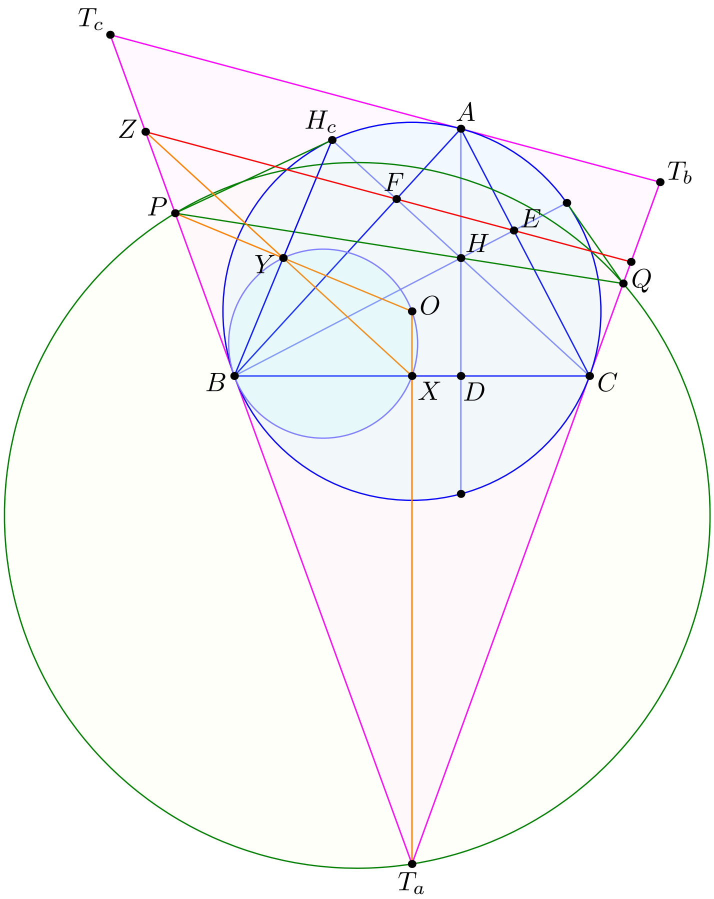
Remark (First claim is faster with complex numbers)**.** It is also straightforward to prove the first claim by using complex numbers. Indeed, in the usual setup, we have that the intersection of the tangents at B and H_c is given explicitly by

$$p = \frac{2b \cdot \left(-\frac{ab}{c}\right)}{b - \frac{ab}{c}} = \frac{2ab}{a - c}$$

and one explicitly checks $p - (a + b + c) \perp (b + c - a)$, as needed.

Claim (Radical axis of ω_A and Γ)**.** Line EF coincides with the radical axis of ω_A and Γ.

Proof. Let lines EF and T_aT_c meet at Z. It suffices to show Z lies on the radical axis, and then repeat the argument on the other side.



Since $\measuredangle FBZ = \measuredangle ABZ = \measuredangle BCA = \measuredangle EFA = \measuredangle ZFB$, it follows $ZB = ZF$. We introduce two other points X and Y on the perpendicular bisector of $\overline{BF}$: they are the midpoints of $\overline{BC}$ and $\overline{BH_c}$.

Since $OX \cdot OT_a = OB^2 = OY \cdot OP$, it follows that $XYPT_a$ is cyclic. Then

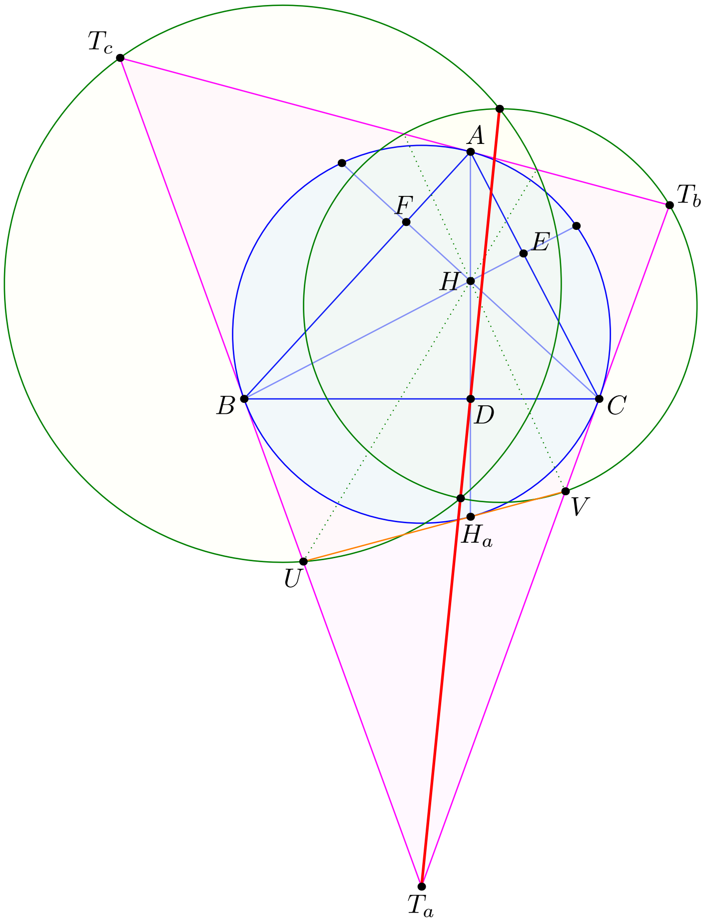
$$ZP \cdot ZT_a = ZX \cdot ZY = ZB^2$$

with the last equality since the circumcircle of $\triangle BXY$ is tangent to Γ (by a $\frac{1}{2}$-homothety at B). So the proof of the claim is complete. $\square$

Finally, we are ready to finish the problem.

Claim. Line DT_a coincides with the radical axis of ω_B and ω_C.

Proof. The point D already coincides with the radical axis because it is the radical center of Γ, ω_B and ω_C. As for the point T_a, we let the tangent to Γ at H_a meet $\overline{T_aT_c}$ at U and V; by the first claim, these lie on ω_C and ω_B respectively.



We need to show $T_aU \cdot T_aT_c = T_aV \cdot T_aT_b$.

But UVT_bT_c is apparently cyclic: the sides $\overline{T_bT_c}$ and $\overline{UV}$ are reflections across a line perpendicular to $\overline{AH_a}$, while the sides $\overline{UT_c}$ and $\overline{VT_b}$ are reflections across a line perpendicular to $\overline{BC}$. So this is true. □

Now since $\triangle DEF$ and $\triangle T_aT_bT_c$ are homothetic (their opposite sides are parallel), and their incenters are respectively H and O, the problem is solved.

Remark (Barycentric approaches with respect to $\triangle T_aT_bT_c$)**.** Because the first claim is so explicit, it is possible to calculate the length of the segment PB. This opens the possibility of using barycentric coordinates with respect to the reference triangle $T_aT_bT_c$, and in fact some contestants were able to complete this approach. Writing $a = T_bT_c$, $b = T_cT_a$, $c = T_aT_b$ one can show that the radical center is the point

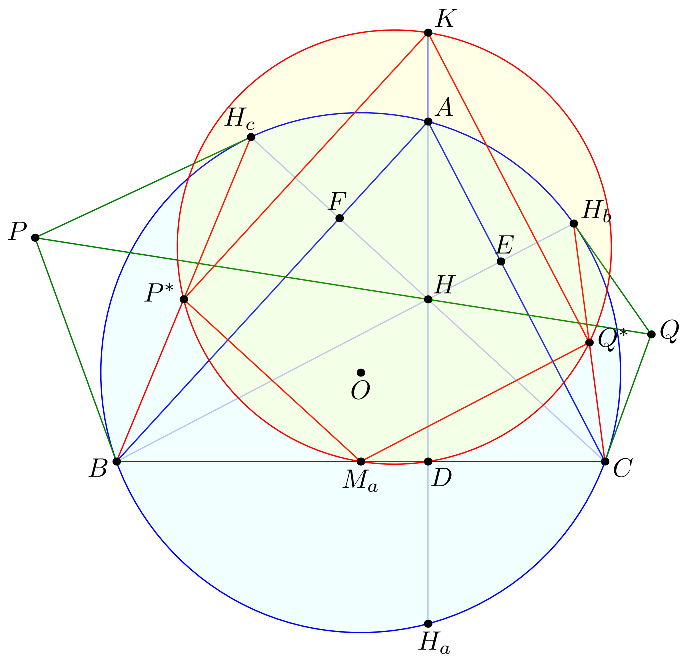
$$\left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c} \right)$$

which is checked to be collinear with the circumcenter and incenter of $\triangle T_aT_bT_c$.

Alternate inversion approach replacing the last two claims, by Serena An. After finding P and Q, it's also possible to solve the problem by using inversion. This eliminates the need to identify line EF as the radical axis of ω_A and Γ.

Inverted points are denoted with $\bullet^*$ as usual, but we will only need two points: P^*, the midpoint of $\overline{BH_c}$, and Q^*, the midpoint of $\overline{CH_b}$. Now, let $M_a = T_a^*$ denote the midpoint $\overline{BC}$ and let K be a point on ray HA with

$$KH = \frac{3}{2}AH.$$



Claim. The points D, P^*, Q^* lie on the circle with diameter $\overline{KM_a}$.

Proof. Consider the homothety at H with scale factor $\frac{3}{2}$. It maps F to the midpoint of $\overline{FH_c}$ and A to K, so we find $\overline{KP^*}$ is the perpendicular bisector of $\overline{FH_c}$. As $\overline{M_aP^*} \parallel \overline{CH}$, we conclude $\angle KP^*M_a = 90°$.

Similarly $\angle KQ^*M_a = 90°$. And $\angle KDM_a = 90°$ is given. □

Claim. Line $\overline{CH}$ coincides with the radical axis of ω_A^* and ω_B^*. In particular, the circles ω_A, ω_B, (COH_c) are coaxial.

Proof. Letting Γ and Γ_9 denote circumcircle and nine-point circle,

$$\begin{aligned} \operatorname{Pow}(C, \omega_A^*) &= CD \cdot CM_a = \operatorname{Pow}(C, \Gamma_9) \\ \operatorname{Pow}(H, \omega_A^*) &= HM \cdot HD = \tfrac{3}{2}HA \cdot \tfrac{1}{2}HH_a = \tfrac{3}{4}\operatorname{Pow}(H, \Gamma). \end{aligned}$$

The same calculation holds with ω_B^*. Now line CH inverts to (COH_c), as needed. □

Since the circles (AOH_a), (BOH_b), (COH_c) have common radical axis equal to line OH, the problem is solved.

Remark (Nikolai Beluhov — generalization with variable ABC and fixed H)**.** Take a fixed circle Γ and a fixed point H in its interior. Then there exist infinitely many triangles ABC with orthocenter H and circumcircle Γ. In fact, for every point A on Γ we get a unique pair of B and C, determined as follows: Let line AH meet Γ again at S_A, and take B and C to be the intersection points of the perpendicular bisector of segment HS_A with Γ.

With this framework, the following generalization is true: The radical center W of ω_A, ω_B, and ω_C is the same point for all such triangles. Indeed, the Euler circle e of triangle ABC is constant because it depends only on H and Γ. Let inversion relative to Γ map e onto Ω. Then all three of T_a, T_b, and T_c lie on Ω, and so, by the solution, W is the homothety center of e and Ω. (We take the homothety center with positive ratio when triangle ABC is acute, and with negative ratio when it is obtuse. When triangle ABC is right-angled, Ω degenerates into a straight line.)

Explicitly, let Γ be the unit circle and put H on the real axis at h. Then W is also on the real axis, at $4h/(h^2 + 3)$.

Furthermore, it turns out the power of W with respect to ω_A, ω_B, and ω_C is constant as well. This, however, is much tougher to prove; we are not aware of a purely geometric proof at this time. Explicitly, in the setting above where Γ is the unit circle and H is on the real axis at h, the power of W with respect to ω_A, ω_B, and ω_C equals $12(h^2 - 1)/(h^2 + 3)^2$.