USEMO 2020/2

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Twitch Solves ISL

Episode 34

Problem

Calvin and Hobbes play a game. First, Hobbes picks a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, 2020\}$, known to both players. Then, Calvin and Hobbes take turns choosing a number from $\{1, 2, \ldots, 2020\}$ which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of $\mathcal{F}$, otherwise Hobbes wins. What is the largest possible size of a family $\mathcal{F}$ that Hobbes could pick while still having a winning strategy?

Video

https://youtu.be/uj93tNL8f7M

Solution

The answer is $4^{1010} - 3^{1010}$. In general, if 2020 is replaced by $2n$, the answer is $4^n - 3^n$.

Construction: The construction is obtained as follows: pair up the numbers as $\{1, 2\}$, $\{3, 4\}$, ..., $\{2019, 2020\}$. Whenever Calvin picks a numbers from one pair, Hobbes elects to pick the other number. Then Calvin can never obtain a subset which has both numbers from one pair. There are indeed $2^{2n} - 3^n$ subsets with this property, so this maximum is achieved.

Bound: The main claim is the following.

Claim. For every k, there are at least $\binom{n}{k}2^k$ sets with k numbers that Calvin can guarantee to obtain after his kth turn.

Proof, due to Andrew Gu. The number of ways that Calvin can choose his first k moves is

$$2n \cdot (2n - 2) \cdot (2n - 4) \cdot \ldots (2n - 2(k - 1)).$$

But each k-element set can be obtained in this way in at most $k!$ ways (based on what order its numbers were taken). So we get a lower bound of

$$\frac{2n \cdot (2n - 2) \cdot (2n - 4) \cdot \ldots (2n - 2(k - 1))}{k!} = 2^k \binom{n}{k}. \qquad \square$$

Thus by summing $k = 0, \ldots, n$ the family S is missing at least $\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n$ subsets, as desired.

Remark (Alternate proof of claim by induction)**.** It is also possible to phrase the proof above using induction on $n + k$. Fix any $a \in \{1, \ldots, 2n\}$; suppose Calvin picks a on the first turn, and Hobbes responds by picking b on the second turn according to his winning strategy. We now have the same game with $\{1, \ldots, 2n\} \setminus \{a, b\}$; so we can apply induction hypothesis with $(n - 1, k - 1)$ and follow through.