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Episode 34

## Problem

Which positive integers can be written in the form

$$
\frac{\operatorname{lcm}(x, y)+\operatorname{lcm}(y, z)}{\operatorname{lcm}(x, z)}
$$

for positive integers $x, y, z$ ?

## Video

https://youtu.be/5a_XCGKiXnI

## External Link

https://aops.com/community/p18471128

## Solution

Let $k$ be the desired value, meaning

$$
-k \operatorname{lcm}(x, z)+\operatorname{lcm}(x, y)+\operatorname{lcm}(y, z)=0 .
$$

Our claim is that the possible values are even integers.
Indeed, if $k$ is even, it is enough to take $(x, y, z)=(1, k / 2,1)$.
For the converse direction we present a few approaches.

First approach using $\nu_{2}$ only. We are going to use the following fact:
Lemma. If $u, v, w$ are nonzero integers with $u+v+w=0$, then either

$$
\begin{aligned}
\nu_{2}(u) & >\nu_{2}(v)=\nu_{2}(w) ; \\
\nu_{2}(v) & >\nu_{2}(w)=\nu_{2}(u) ; \quad \text { or } \\
\nu_{2}(w) & >\nu_{2}(u)=\nu_{2}(v)
\end{aligned}
$$

Proof. Let's assume WLOG that $e=\nu_{2}(w)$ is minimal. If both $\nu_{2}(u)$ and $\nu_{2}(v)$ are strictly greater than $e$, then $\nu_{2}(u+v+w)=e$ which is impossible. So assume WLOG again that $\nu_{2}(v)=\nu_{2}(w)=e$. Then

$$
u=-\left(2^{e} \cdot \text { odd }+2^{e} \cdot \text { odd }\right)=-2^{e} \cdot \text { even }
$$

so $\nu_{2}(u) \geq e+1$.
However, if we assume for contradiction that $k$ is odd, then

$$
\begin{aligned}
\nu_{2}(-k \operatorname{lcm}(x, z)) & =\max \left(\nu_{2}(x), \nu_{2}(z)\right) \\
\nu_{2}(\operatorname{lcm}(x, y)) & =\max \left(\nu_{2}(x), \nu_{2}(y)\right) \\
\nu_{2}(\operatorname{lcm}(y, z)) & =\max \left(\nu_{2}(y), \nu_{2}(z)\right) .
\end{aligned}
$$

In particular, the largest two numbers among the three right-hand sides must be equal. So by the lemma, there is no way the three numbers $(-k \operatorname{lcm}(x, z), \operatorname{lcm}(x, y), \operatorname{lcm}(y, z))$ could have sum zero.

Second approach using $\nu_{p}$ for general $p$. We'll prove the following much stronger claim (which will obviously imply $k$ is even).

Claim. We must have $\operatorname{lcm}(x, z) \mid \operatorname{lcm}(x, y)=\operatorname{lcm}(y, z)$.
Proof. Take any prime $p$ and look at three numbers $\nu_{p}(x), \nu_{p}(y), \nu_{p}(z)$. We'll show that

$$
\max \left(\nu_{p}(x), \nu_{p}(z)\right) \leq \max \left(\nu_{p}(x), \nu_{p}(y)\right)=\max \left(\nu_{p}(y), \nu_{p}(z)\right)
$$

If $\nu_{p}(y)$ is the (non-strict) maximum, then the claim is obviously true.
If not, by symmetry assume WLOG that $\nu_{p}(x)$ is largest, so that $\nu_{p}(x)>\nu_{p}(y)$ and $\nu_{p}(x) \geq \nu_{p}(z)$. However, from the given equation, we now have $\nu_{p}(\operatorname{lcm}(y, z)) \geq \nu_{p}(x)$. This can only occur if $\nu_{p}(z)=\nu_{p}(x)$. So the claim is true in this case too.

Third approach without taking primes (by circlethm). By scaling, we may as well assume $\operatorname{gcd}(x, y, z)=1$.

Let $d_{x y}=\operatorname{gcd}(x, y)$, etc. Now note that $\operatorname{gcd}\left(d_{x y}, d_{x z}\right)=1$, and cyclically! This allows us to write the following decomposition:

$$
\begin{aligned}
& x=d_{x y} d_{x z} a \\
& y=d_{x y} d_{y z} b \\
& z=d_{x z} d_{y z} c .
\end{aligned}
$$

We also have $\operatorname{gcd}(a, b)=\operatorname{gcd}(b, c)=\operatorname{gcd}(c, a)=1$ now.
Now, we have

$$
\begin{aligned}
\operatorname{lcm}(x, y) & =d_{x y} d_{x z} d_{y z} a b \\
\operatorname{lcm}(y, z) & =d_{x y} d_{x z} d_{y z} b c \\
\operatorname{lcm}(x, z) & =d_{x y} d_{x z} d_{y z} a c
\end{aligned}
$$

and so substituting this in to the equation gives

$$
k=b \cdot\left(\frac{1}{a}+\frac{1}{c}\right)
$$

For $a, b, c$ coprime this can only be an integer if $a=c$, so $k=2 b$.
Remark. From $a=c=1$, the third approach also gets the nice result that $\operatorname{lcm}(x, y)=$ $\operatorname{lcm}(y, z)$ in the original equation.

