USEMO 2020/1

Evan Chen

Twitch Solves ISL

Episode 34

Problem

Which positive integers can be written in the form

$$\frac{\operatorname{lcm}(x, y) + \operatorname{lcm}(y, z)}{\operatorname{lcm}(x, z)}$$

for positive integers x, y, z?

Video

https://youtu.be/5a_XCGKiXnI

External Link

https://aops.com/community/p18471128

Solution

Let k be the desired value, meaning

$$-k \operatorname{lcm}(x,z) + \operatorname{lcm}(x,y) + \operatorname{lcm}(y,z) = 0.$$

Our claim is that the possible values are even integers.

Indeed, if k is even, it is enough to take $(x,y,z) = (1, k/2, 1)$.

For the converse direction we present a few approaches.

First approach using ν_2 only. We are going to use the following fact:

Lemma. If u, v, w are nonzero integers with $u+v+w=0$, then either

$$\begin{aligned} \nu_2(u) &> \nu_2(v) = \nu_2(w); \\ \nu_2(v) &> \nu_2(w) = \nu_2(u); \quad \text{or} \\ \nu_2(w) &> \nu_2(u) = \nu_2(v). \end{aligned}$$

Proof. Let's assume WLOG that $e = \nu_2(w)$ is minimal. If both $\nu_2(u)$ and $\nu_2(v)$ are strictly greater than e, then $\nu_2(u+v+w) = e$ which is impossible. So assume WLOG again that $\nu_2(v) = \nu_2(w) = e$. Then

$$u = -(2^e \cdot \text{odd} + 2^e \cdot \text{odd}) = -2^e \cdot \text{even}$$

so $\nu_2(u) \ge e+1$. □

However, if we assume for contradiction that k is odd, then

$$\begin{aligned} \nu_2(-k \operatorname{lcm}(x,z)) &= \max(\nu_2(x), \nu_2(z)) \\ \nu_2(\operatorname{lcm}(x,y)) &= \max(\nu_2(x), \nu_2(y)) \\ \nu_2(\operatorname{lcm}(y,z)) &= \max(\nu_2(y), \nu_2(z)). \end{aligned}$$

In particular, the *largest* two numbers among the three right-hand sides must be equal. So by the lemma, there is no way the three numbers $(-k \operatorname{lcm}(x,z), \operatorname{lcm}(x,y), \operatorname{lcm}(y,z))$ could have sum zero.

Second approach using ν_p for general p. We'll prove the following much stronger claim (which will obviously imply k is even).

Claim. We must have $\operatorname{lcm}(x,z) \mid \operatorname{lcm}(x,y) = \operatorname{lcm}(y,z)$.

Proof. Take any prime p and look at three numbers $\nu_p(x)$, $\nu_p(y)$, $\nu_p(z)$. We'll show that

$$\max(\nu_p(x), \nu_p(z)) \le \max(\nu_p(x), \nu_p(y)) = \max(\nu_p(y), \nu_p(z)).$$

If $\nu_p(y)$ is the (non-strict) maximum, then the claim is obviously true.

If not, by symmetry assume WLOG that $\nu_p(x)$ is largest, so that $\nu_p(x) > \nu_p(y)$ and $\nu_p(x) \ge \nu_p(z)$. However, from the given equation, we now have $\nu_p(\operatorname{lcm}(y,z)) \ge \nu_p(x)$. This can only occur if $\nu_p(z) = \nu_p(x)$. So the claim is true in this case too. □

Third approach without taking primes (by `circlethm`). By scaling, we may as well assume $\gcd(x, y, z) = 1$.

Let $d_{xy} = \gcd(x, y)$, etc. Now note that $\gcd(d_{xy}, d_{xz}) = 1$, and cyclically! This allows us to write the following decomposition:

$$\begin{aligned} x &= d_{xy} d_{xz} a \\ y &= d_{xy} d_{yz} b \\ z &= d_{xz} d_{yz} c. \end{aligned}$$

We also have $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$ now.

Now, we have

$$\begin{aligned} \operatorname{lcm}(x, y) &= d_{xy} d_{xz} d_{yz} ab \\ \operatorname{lcm}(y, z) &= d_{xy} d_{xz} d_{yz} bc \\ \operatorname{lcm}(x, z) &= d_{xy} d_{xz} d_{yz} ac \end{aligned}$$

and so substituting this in to the equation gives

$$k = b \cdot \left(\frac{1}{a} + \frac{1}{c} \right).$$

For a, b, c coprime this can only be an integer if $a = c$, so $k = 2b$.

Remark. From $a = c = 1$, the third approach also gets the nice result that $\operatorname{lcm}(x, y) = \operatorname{lcm}(y, z)$ in the original equation.