

Twitch 033.4

Evan Chen

TWITCH SOLVES ISL

Episode 33

Problem

For each positive integer n , simplify

$$\sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2n}{2k}}.$$

Video

<https://youtu.be/aYeJxgRJeYw>

Solution

We have

$$\frac{\binom{n}{k}^2}{\binom{2n}{2k}} = \frac{(2k)!(2n-2k)!n!n!}{(2n)!k!(n-k)!k!(n-k)!} = \frac{\binom{2k}{k}\binom{2n-2k}{n-k}}{\binom{2n}{n}}$$

Claim (Famous identity).

$$\sum_{a+b=n} \binom{2a}{a} \binom{2b}{b} = 4^n$$

Proof. Square both sides of the identity

$$\frac{1}{\sqrt{1-4X}} = \sum_{k \geq 0} \binom{2k}{k} X^k. \quad \square$$

Consequently, the final answer is

$$\frac{4^n}{\binom{2n}{n}}.$$