Twitch 033.4

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Episode 33

Problem

For each positive integer n, simplify

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{2k}}.$$

Video

https://youtu.be/aYeJxgRJeYw

Solution

We have

$$\frac{\binom{n}{k}^2}{\binom{2n}{2k}} = \frac{(2k)!(2n-2k)!n!n!}{(2n)!k!(n-k)!k!(n-k)!} = \frac{\binom{2k}{k}\binom{2n-2k}{n-k}}{\binom{2n}{n}}$$

Claim (Famous identity)**.**

$$\sum_{a+b=n} \binom{2a}{a}\binom{2b}{b} = 4^n$$

Proof. Square both sides of the identity

$$\frac{1}{\sqrt{1-4X}} = \sum_{k\geq 0} \binom{2k}{k} X^k. \qquad \square$$

Consequently, the final answer is

$$\frac{4^n}{\binom{2n}{n}}.$$