

# Twitch 033.1

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TWITCH SOLVES ISL

Episode 33

## Problem

Let  $H$  be the orthocenter of acute triangle  $\triangle ABC$ . Two points  $X$  and  $Y$  are on the circumcircle of triangle  $\triangle ABC$  such that  $H$  lies on chord  $XY$ . Let  $P$  and  $Q$  be the feet of the altitudes from  $H$  onto  $AX$  and  $AY$ , respectively, and let line  $PQ$  intersect line  $XY$  at  $T$ . Let  $E$  and  $F$  be the feet of the altitudes from  $B$  and  $C$  to  $AC$  and  $AB$ .

Prove that as the chord  $XY$  containing  $H$  varies, point  $T$  traces out part of a circle whose center lies on line  $EF$ .

## Video

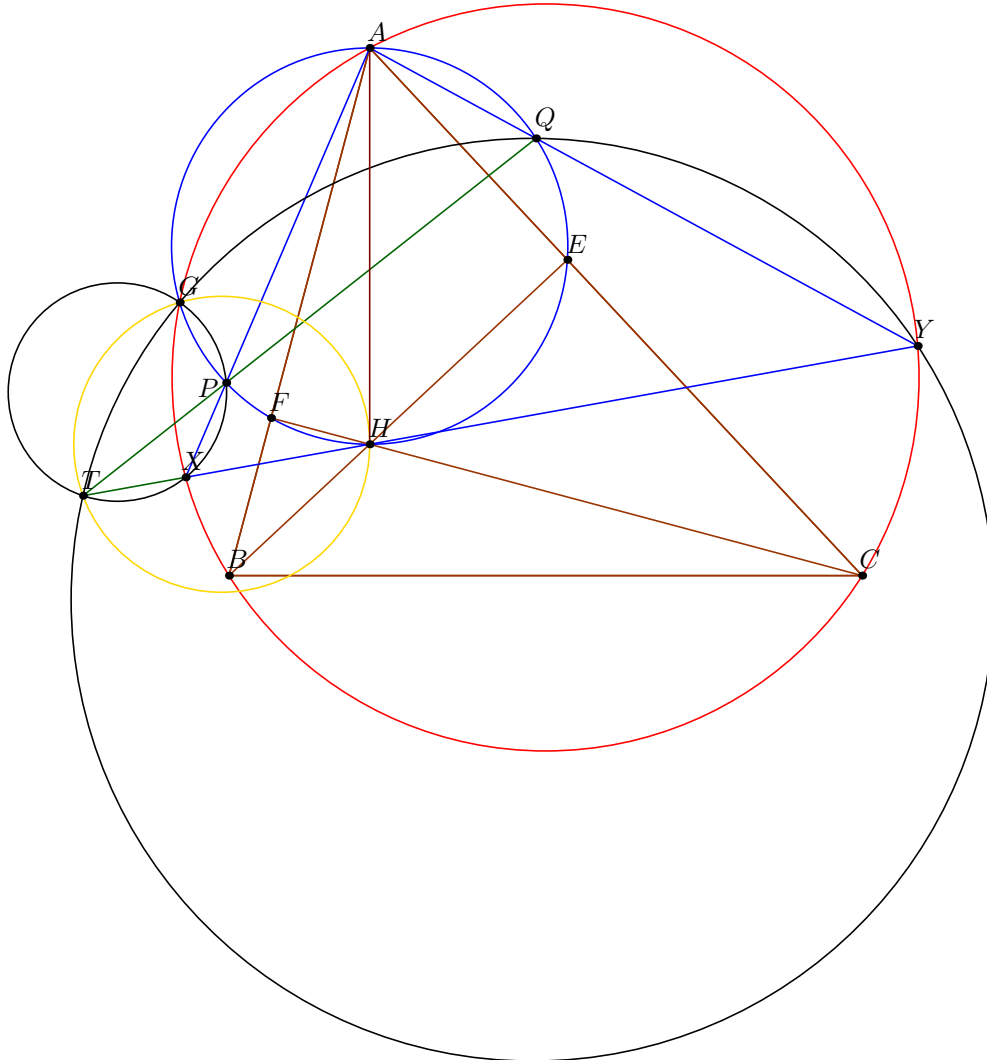
<https://youtu.be/zMN8WeefUuI>

### Solution

We introduce  $G = (AEF) \cap (ABC) \neq A$ .

**Claim.** The point  $G$  is the Miquel point of complete quadrilateral  $XPQY$ .

*Proof.* Clear. □



By this point, we already have

$$\angle GTH = \angle GQY = \angle GQY = \angle GHA$$

which is fixed as the chord varies. This already shows  $T$  lies on a fixed circle and we need only to show its center lies on line  $EF$ .

In fact, the conditions imply that  $(GHT)$  is orthogonal to  $(AEF)$ . Now we prove that:

**Claim.**  $(GH; FE) = -1$ .

*Proof.* Since  $GF/GE = BH/HC = HF/HE$  by spiral similarity. (Alternatively, note that  $GH, FF, EE$  bisect  $BC$ .) □

The harmonic quadrilateral, together with the orthogonality, imply that  $(GHT)$  is exactly the Apollonian circle through  $G$  and  $H$  with respect to  $E$  and  $F$ , so its center lies on line  $EF$ .