Twitch 033.1

Evan Chen

Twitch Solves ISL

Episode 33

Problem

Let H be the orthocenter of acute triangle $\triangle ABC$. Two points X and Y are on the circumcircle of triangle $\triangle ABC$ such that H lies on chord XY. Let P and Q be the feet of the altitudes from H onto AX and AY, respectively, and let line PQ intersect line XY at T. Let E and F are the feet of the altitudes from B and C to AC and AB.

Prove that as the chord XY containing H varies, point T traces out part of a circle whose center lies on line EF.

Video

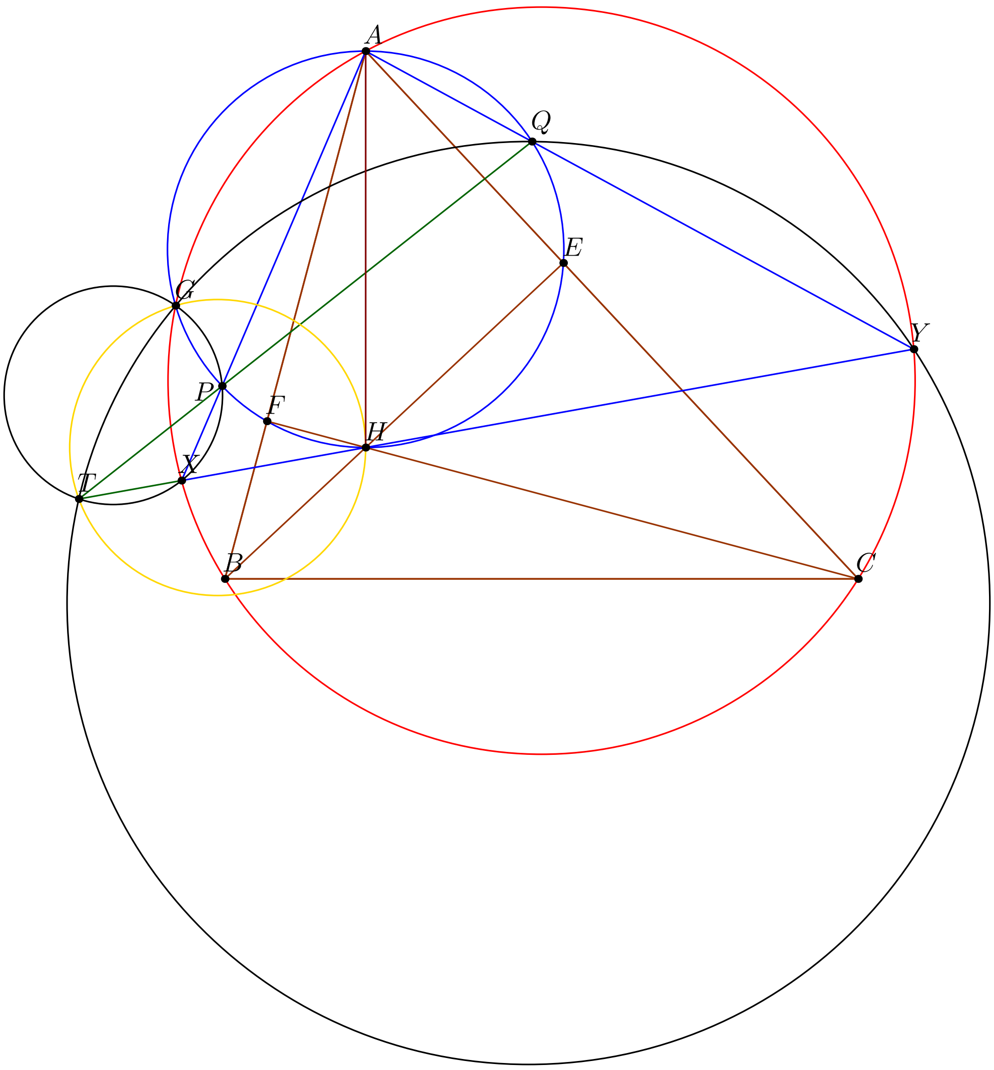
https://youtu.be/zMN8WeefUuI

Solution

We introduce $G = (AEF) \cap (ABC) \neq A$.

Claim. The point G is the Miquel point of complete quadrilateral $XPQY$.

Proof. Clear. □



By this point, we already have

$$\measuredangle GTH = \measuredangle GQY = \measuredangle GQY = \measuredangle GHA$$

which is fixed as the chord varies. This already shows T lies on a fixed circle and we need only to show its center lies on line EF.

In fact, the conditions imply that (GHT) is orthogonal to (AEF). Now we prove that:

Claim. $(GH; FE) = -1$.

Proof. Since $GF/GE = FB/EC = HF/HE$ by spiral similarity. (Alternatively, note that GH, FF, EE bisect BC.) □

The harmonic quadrilateral, together with the orthogonality, imply that (GHT) is exactly the Apollonian circle through G and H with respect to E and F, so its center lies on line EF.