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Twitch Solves ISL

Episode 33

Problem

Find the number of subsets $A \subset M = \{2^0, 2^1, 2^2, \dots, 2^{2005}\}$ such that equation $x^2 - S(A)x + S(B) = 0$ has integral roots, where $S(M)$ is the sum of all elements of M, $B = M \setminus A$, and A and B are not empty.

Video

https://youtu.be/L9O1f7FP8bs

External Link

https://aops.com/community/p220242

Solution

Let $s = S(A)$, which in binary may be any number between $2^0 = 1$ and $2^1 + 2^2 + \cdots + 2^{2005} = 2^{2006} - 2$. The quadratic has integer roots if and only if the discriminant is a perfect square, and it equals

$$\Delta^2 = s^2 - 4\left(2^{2006} - 1 - s\right) = s^2 + 4s + 4 - 4 \cdot 2^{2006}.$$

We can rewrite this as

$$(s+2)^2 - \Delta^2 = 2^{2008}.$$

The difference of squares gives us now that s should be of the form

$$s = \frac{2^n + 2^{2008-n}}{2} - 2.$$

Keeping in mind that $1 \leq s < 2^{2006} - 1$, we have valid (different) solutions $n = 2, 3, 4, 5, \ldots, 1004$. So there are 1003 valid solutions.