APMO 2009/1

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Twitch Solves ISL

Episode 33

Problem

Consider the following operation on positive real numbers written on a blackboard: Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

Video

https://youtu.be/uj93tNL8f7M

Solution

The problem follows from one observation:

Claim (Main claim)**.** The sum of the squares of reciprocals does not change under this operation.

Proof. Clear. □

Hence, if $x_1, \dots, x_n$ are the numbers on the board, when $n = k^2$, they satisfy

$$\frac{1}{r^2} = \sum \frac{1}{x_n^2}$$

which implies $\min(x_1, \dots, x_n) \le \sqrt{n}r = kr$.