# ToT Fall 1999 J-A6 <br> Evan Chen 

## Twitch Solves ISL

Episode 32

## Problem

A rook is allowed to move one cell either horizontally or vertically. After 64 moves the rook visited all cells of the $8 \times 8$ chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction cannot be equal.

## Video

https://youtu.be/JMNCvAb0CgA

## External Link

https://aops.com/community/p15168247

## Solution

Identify the cells of the board as points of an equally spaced grid. We view the path of the rook as a closed cycle $\mathcal{P}$ which is naturally a (simple) polygon (but possibly non-convex) whose interior angles are either $90^{\circ}$ or $270^{\circ}$. By the Jordan curve theorem, the polygon has an interior.

We will perform operations which

- gradually increase the area of $\mathcal{P}$,
- while maintaining the property that all 64 points either remain on $\mathcal{P}$ or inside it.

The main idea is the following foothold.
Claim. If the properties above hold then either $\mathcal{P}$ is a rectangle or there exists two consecutive $270^{\circ}$ angles of $\mathcal{P}$ which are one unit apart (a "U-shape").

Proof. Suppose that $\mathcal{P}$ has no U-shapes. We exhibit an injective map from every $270^{\circ}$ angle of $\mathcal{P}$ to some $90^{\circ}$ angle of $\mathcal{P}$.

Take any $270^{\circ}$ angle with vertex at $X$, as shown below, and look at the point $Y$ marked in red in the figure below (assume without loss of generality the orientation shown). Because there are no U-turns, we are assuming $Y$ is But we are assuming no lattice points are outside $\mathcal{P}$, so $Y$ is part of $\mathcal{P}$ later; but then it is also is part of an angle of $\mathcal{P}$.


Since $\mathcal{P}$ has a well-defined interior and exterior, and the interior is to the southwest of $X$, it follows the interior of $\mathcal{P}$ is to the northeast of $Y$. So we take the map $X \mapsto Y$. It is obviously injective and this completes the proof.

Since the total sum of angle measures is $180^{\circ} \cdot(n-2)$ where $n$ is the number of sides of $\mathcal{P}$, this will imply $n \leq 4$, i.e. that $n$ is a rectangle. In other words, the only polygons that have no U-shape are rectangles.

Given a U-shape as described, we apply an operation to add the "missing" square into $\mathcal{P}$ as shown below. Doing this preserves all the condition and leaves a "green domino" (two adjacent cells); so we apply the operation until we get a rectangle.


Since no points lie outside the rectangle, the final rectangle must be the outer perimeter of the grid; in other words, the final configuration gives a green domino tiling of the interior $6 \times 6$ lattice, as shown.


Since the orientation of the dominoes is opposite the orientation of the two "deleted" segments when it was created, the problem is reduced to proving the following claim.

Claim. A domino tiling of a $6 \times 6$ board has an unequal number of horizontal and vertical dominoes.

Proof. There are 18 dominoes and we claim there are an even number of each orientation. Indeed, color the odd-numbered rows; there are 18 shaded cells now. A vertical domino covers an odd number of shaded cells, while a horizontal domino covers an even number of shaded cells. So there are an even number of vertical dominoes, as needed.

Remark. Here is a counterexample for an $10 \times 10$ board which generalizes easily, so the problem is false in general if 8 is replaced by a $2 \bmod 4$ number.


