ToT Fall 1999 J-A6

Evan Chen

Twitch Solves ISL

Episode 32

Problem

A rook is allowed to move one cell either horizontally or vertically. After 64 moves the rook visited all cells of the 8×8 chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction cannot be equal.

Video

https://youtu.be/JMNCvAb0CgA

Solution

Identify the cells of the board as points of an equally spaced grid. We view the path of the rook as a closed cycle $\mathcal{P}$ which is naturally a (simple) polygon (but possibly non-convex) whose interior angles are either 90° or 270°. By the Jordan curve theorem, the polygon has an interior.

We will perform operations which

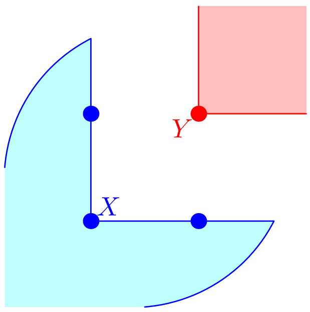
- gradually increase the area of $\mathcal{P}$,
- while maintaining the property that all 64 points either remain on $\mathcal{P}$ or inside it.

The main idea is the following foothold.

Claim. If the properties above hold then either $\mathcal{P}$ is a rectangle or there exists two consecutive 270° angles of $\mathcal{P}$ which are one unit apart (a "U-shape").

Proof. Suppose that $\mathcal{P}$ has no U-shapes. We exhibit an injective map from every 270° angle of $\mathcal{P}$ to some 90° angle of $\mathcal{P}$.

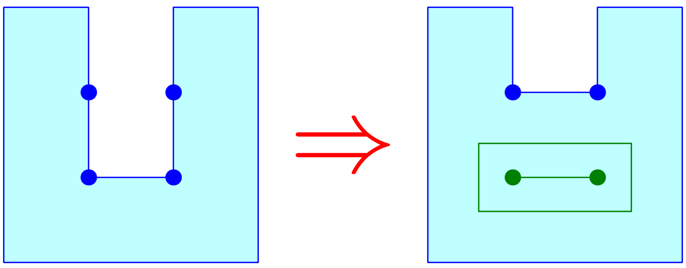
Take any 270° angle with vertex at X, as shown below, and look at the point Y marked in red in the figure below (assume without loss of generality the orientation shown). Because there are no U-turns, we are assuming Y is But we are assuming no lattice points are outside $\mathcal{P}$, so Y is part of $\mathcal{P}$ later; but then it is also is part of an angle of $\mathcal{P}$.



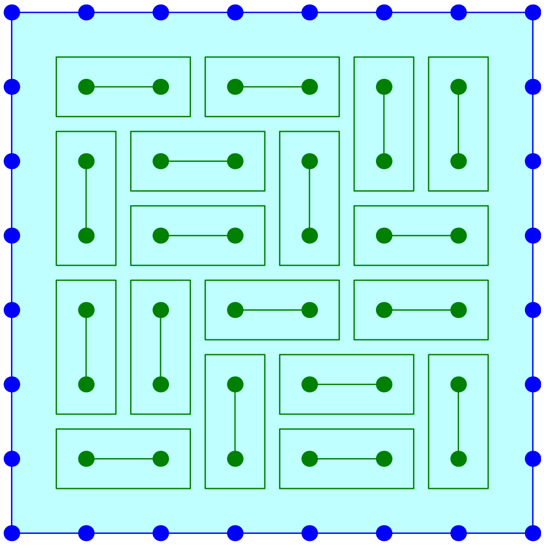
Since $\mathcal{P}$ has a well-defined interior and exterior, and the interior is to the southwest of X, it follows the interior of $\mathcal{P}$ is to the northeast of Y. So we take the map $X \mapsto Y$. It is obviously injective and this completes the proof.

Since the total sum of angle measures is $180^\circ \cdot (n-2)$ where n is the number of sides of $\mathcal{P}$, this will imply $n \le 4$, i.e. that n is a rectangle. In other words, the only polygons that have no U-shape are rectangles. $\square$

Given a U-shape as described, we apply an operation to add the "missing" square into $\mathcal{P}$ as shown below. Doing this preserves all the condition and leaves a "green domino" (two adjacent cells); so we apply the operation until we get a rectangle.



Since no points lie outside the rectangle, the final rectangle must be the outer perimeter of the grid; in other words, the final configuration gives a green domino tiling of the interior 6×6 lattice, as shown.



Since the orientation of the dominoes is opposite the orientation of the two "deleted" segments when it was created, the problem is reduced to proving the following claim.

Claim. A domino tiling of a 6×6 board has an unequal number of horizontal and vertical dominoes.

Proof. There are 18 dominoes and we claim there are an even number of each orientation. Indeed, color the odd-numbered rows; there are 18 shaded cells now. A vertical domino covers an odd number of shaded cells, while a horizontal domino covers an even number of shaded cells. So there are an even number of vertical dominoes, as needed. □

Remark. Here is a counterexample for an 10×10 board which generalizes easily, so the problem is false in general if 8 is replaced by a 2 mod 4 number.