Cono Sur 2011/6

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Twitch Solves ISL

Episode 32

Problem

Let Q be a $(2n+1) \times (2n+1)$ board. Some of its cells are colored black in such a way that every 2×2 board of Q has at most 2 black cells. Find the maximum amount of black cells that the board may have.

Video

https://youtu.be/S6AYLVuUzn8

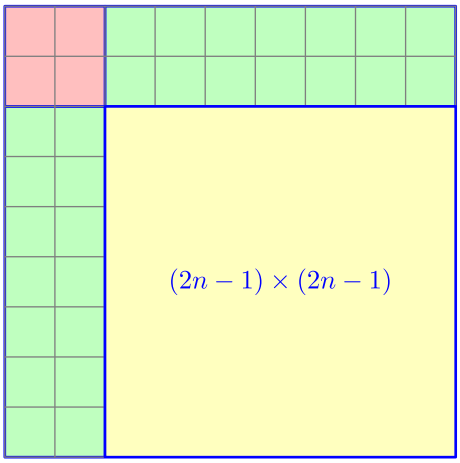
External Link

https://aops.com/community/p3583778

Solution

The answer is $(n+1)(2n+1)$ achieved by coloring all the odd numbered rows.

We proceed by induction on n with the base case $n=0$ being clear. For the other direction, we are going to decompose our $(2n+1)\times(2n+1)$ board into a 2×2 square, two $2\times(2n-1)$ strips, and a $(2n-1)\times(2n-1)$ square, as shown below.



To get the bound, we observe that:

- The red 2×2 square has at most two black cells.
- The western green rectangle has at most $2n$ black cells, with equality if and only if the rows of length 2 are shaded alternating black. The same is true for the northern green rectangle.
- However, equality cannot simultaneously occur: if the two green rectangles both have $2n$ black cells, then the red 2×2 square has at most one black cell (at the northwestern corner).

Hence, outside the yellow square, there are at most $(2n+2n+2)-1=4n+1$ black squares.

Now, induction gives a bound of

$$\underbrace{n\cdot(2n-1)}_{\text{by IH}}+\underbrace{(4n+1)}_{\text{green/red}}=(n+1)(2n+1).$$