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Episode 31

Problem

Find the number of integers c such that $-2007 \leq c \leq 2007$ and there exists an integer x such that $x^2 + c$ is a multiple of 2^{2007}.

Video

https://youtu.be/AzYdaQk1TmM

External Link

https://aops.com/community/p957989

Solution

The main claim is the following, which can be proven:

Claim. The odd squares mod 2^k are exactly the $1 \pmod 8$ numbers for $k \geq 3$.

Proof. For $k \geq 3$, it is known that the number 5 has order 2^{k-2} modulo 2^k. The powers of 5 cover all the $1 \pmod 4$ numbers exactly, and the even powers of 5 give all $1 \pmod 8$. This shows that every $1 \pmod 8$ number is a square.

On the other hand, every square is obviously $1 \pmod 8$. $\square$

Remark. The seeming choice of a magic number 5 is actually just for concreteness. The numbers 3, 13, 99 would work equally well, and in general I think any odd number g with $\nu_2(g^2 - 1) = 3$ should work.

Let $n = -c$. Hence we want the number of n such that $\nu_2(n)$ is even and the odd part is $1 \pmod 8$. This is an annoying casework calculation

- $n = 0$: 1 case
- $\nu_2(n) = 0$: $\{-2007, \dots, -15, -7, 1, 9, \dots, 2001\}$ gives 502 numbers
- $\nu_2(n) = 2$: $\{4 \cdot -495 = -1980, \dots, 4 \cdot 497 = 1988, \}$ gives 125 numbers
- $\nu_2(n) = 4$: $\{16 \cdot -119 = -1904, \dots, 16 \cdot 121 = 1936\}$ gives 31 numbers
- $\nu_2(n) = 6$: $\{64 \cdot -31 = -1984, \dots, 64 \cdot 25 = 1600\}$ gives 8 numbers
- $\nu_2(n) = 8$: $\{256 \cdot -7 = -1792, 256 \cdot 1 = 256\}$ gives 2 numbers
- $\nu_2(n) = 10$: $\{1024\}$ gives 1 number.

The total count is 670.