IMO 2020/6

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Twitch Solves ISL

Episode 30

Problem

Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. Prove there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $\Omega(n^{-1/3})$.

(A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

Video

https://youtu.be/OxeeSsdEgwI

Solution

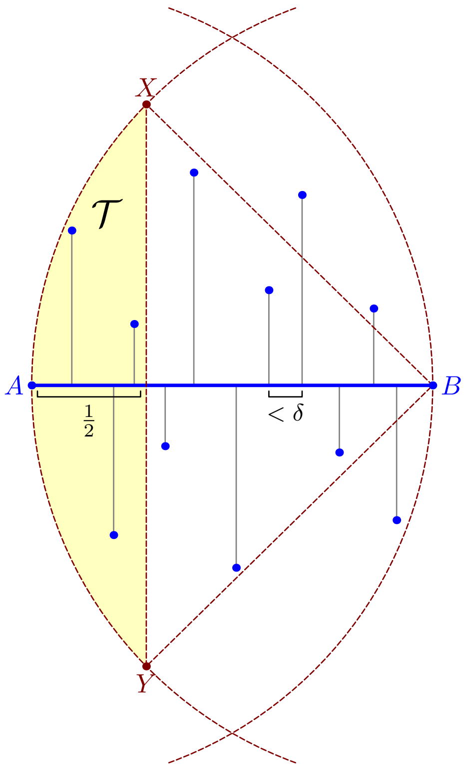
We present the official solution given by the Problem Selection Committee.

Let's suppose that among all projections of points in $\mathcal{S}$ onto some line m, the maximum possible distance between two consecutive projections is δ. We will prove that $\delta \geq \Omega(n^{-1/3})$, solving the problem.

We make the following the definitions:

- Define A and B as the two points farthest apart in $\mathcal{S}$. This means that all points lie in the intersections of the circles centered at A and B with radius $R = AB \geq 1$.
- We pick chord $\overline{XY}$ of $\odot(B)$ such that $\overline{XY} \perp \overline{AB}$ and the distance from A to $\overline{XY}$ is exactly $\frac{1}{2}$.
- We denote by $\mathcal{T}$ the smaller region bound by $\odot(B)$ and chord $\overline{XY}$.

The figure is shown below with $\mathcal{T}$ drawn in yellow, and points of $\mathcal{S}$ drawn in blue.



Claim (Length of AB + Pythagorean theorem)**.** We have $XY < 2\sqrt{n\delta}$.

Proof. First, note that we have $R = AB < (n-1) \cdot \delta$, since the n projections of points onto AB are spaced at most δ apart. The Pythagorean theorem gives

$$XY = 2\sqrt{R^2 - \left(R - \frac{1}{2}\right)^2} = 2\sqrt{R - \frac{1}{4}} < 2\sqrt{n\delta}. \qquad \square$$

Claim ($|\mathcal{T}|$ lower bound + narrowness)**.** We have $XY > \frac{\sqrt{3}}{2}\left(\frac{1}{2}\delta^{-1} - 1\right)$.

Proof. Because $\mathcal{T}$ is so narrow (has width $\frac{1}{2}$ only), the projections of points in $\mathcal{T}$ onto line XY are spaced at least $\frac{\sqrt{3}}{2}$ apart (more than just δ). This means

$$XY > \frac{\sqrt{3}}{2}\left(|\mathcal{T}| - 1\right).$$

But projections of points in $\mathcal{T}$ onto the segment of length $\frac{1}{2}$ are spaced at most δ apart, so apparently

$$|\mathcal{T}| > \frac{1}{2} \cdot \delta^{-1}.$$

This implies the result. $\square$

Combining these two this implies $\delta \geq \Omega(n^{-1/3})$ as needed.

Remark. The constant $1/3$ in the problem is actually optimal and cannot be improved; the constructions give an example showing $\Theta(n^{-1/3} \log n)$.