

# IMO 2020/3

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TWITCH SOLVES ISL

Episode 30

## Problem

There are  $4n$  pebbles of weights  $1, 2, 3, \dots, 4n$ . Each pebble is coloured in one of  $n$  colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles the total weights of both piles are the same, and each pile contains two pebbles of each colour.

## Video

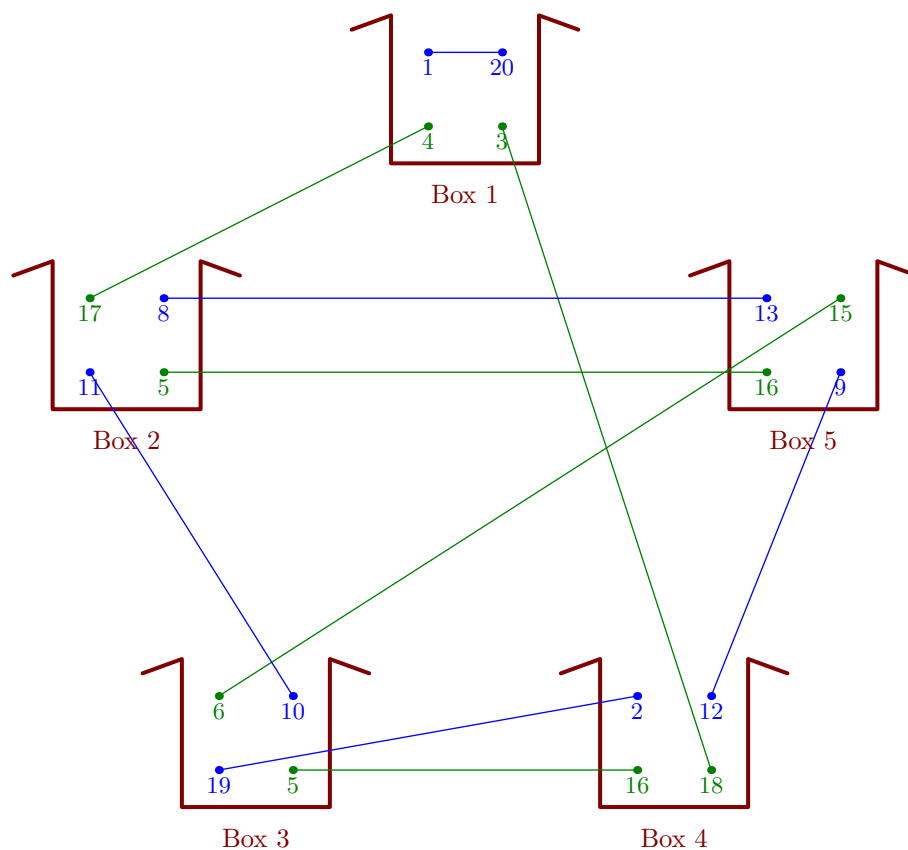
<https://youtu.be/0xeeSsdEgwI>

## Solution

The first key idea is the deep fact that

$$1 + 4n = 2 + (4n - 1) = 3 + (4n - 2) = \dots$$

So, place all four pebbles of the same colour in a box (hence  $n$  boxes). For each  $k = 1, 2, \dots, 2n$  we tape a piece of string between pebble  $k$  and  $4n + 1 - k$ . To solve the problem, it suffices to paint each string either blue or green such that each box has two blue strings and two green strings (where a string between two pebbles in the same box counts double).



We can therefore rephrase the problem as follows, if we view boxes as vertices and strings as edges:

**Claim.** Given a 4-regular multigraph on  $n$  vertices (where self-loops are allowed and have degree 2), one can color the edges blue and green such that each vertex has two blue and two green edges.

*Proof.* Each connected component of the graph can be decomposed into an Eulerian circuit, since 4 is even. A connected component with  $k$  vertices has  $2k$  edges in its Eulerian circuit, so we may color the edges in this circuit alternating green and blue. This may be checked to work.  $\square$