

# IMO 2020/2

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TWITCH SOLVES ISL

Episode 30

## Problem

The real numbers  $a, b, c, d$  are such that  $a \geq b \geq c \geq d > 0$  and  $a + b + c + d = 1$ . Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

## Video

<https://youtu.be/JfRrlvbzKHk>

## External Link

<https://aops.com/community/p17821569>

## Solution

By weighted AM-GM we have

$$a^a b^b c^c d^d \leq \sum_{\text{cyc}} \frac{a}{a+b+c+d} \cdot a = a^2 + b^2 + c^2 + d^2.$$

Consequently, it is enough to prove that

$$(a^2 + b^2 + c^2 + d^2)(a + 2b + 3c + 4d) \leq 1 = (a + b + c + d)^3.$$

Expand both sides to get

$$\begin{array}{cccccccc} +a^3 & +b^2a & +c^2a & +d^2a & & +a^3 & +3b^2a & +3c^2a & +3d^2a \\ +2a^2b & +2b^3 & +2bc^2 & +2d^2b & & +3a^2b & +b^3 & +3bc^2 & +3d^2b \\ +3a^2c & +3b^2c & +3c^3 & +3d^2c & < & +3a^2c & +3b^2c & +c^3 & +3d^2c \\ +4a^2d & +4b^2d & +4c^2d & +4d^3 & & +3a^2d & +3b^2d & +3c^2d & +d^3 \\ & & & & & +6abc & +6bcd & +6cda & +6dab \end{array}$$

In other words, we need to prove that

$$\begin{array}{cccccccc} & & & & & +2b^2a & +2c^2a & +2d^2a \\ & & & & +a^2b & & +bc^2 & +d^2b \\ +b^3 & & & & < & & & \\ & +2c^3 & & & & & & \\ +a^2d & +b^2d & +c^2d & +3d^3 & & +6abc & +6bcd & +6cda & +6dab \end{array}$$

This follows since

$$\begin{aligned} 2b^2a &\geq b^3 + b^2d \\ 2c^2a &\geq 2c^3 \\ 2d^2a &\geq 2d^3 \\ a^2b &\geq a^2d \\ bc^2 &\geq c^2d \\ d^2b &\geq d^3 \end{aligned}$$

and  $6(abc + bcd + cda + dab) > 0$ .

**Remark.** Fedor Petrov provides the following motivational comments for why the existence of this solution is not surprising:

Better to think about mathematics. You have to bound from above a product  $(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2)$ , the coefficients 1, 2, 3, 4 are increasing and so play on your side, so plausibly  $(a + b + c + d)^3$  should majorize this term-wise, you check it and this appears to be true.