IMO 2020/2

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Twitch Solves ISL

Episode 30

Problem

The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

Video

https://youtu.be/JfRrlvbzKHk

External Link

https://aops.com/community/p17821569

Solution

By weighted AM-GM we have

$$a^a b^b c^c d^d \le \sum_{\text{cyc}} \frac{a}{a+b+c+d} \cdot a = a^2+b^2+c^2+d^2.$$

Consequently, it is enough to prove that

$$(a^2+b^2+c^2+d^2)(a+2b+3c+4d) \le 1 = (a+b+c+d)^3.$$

Expand both sides to get

$$\begin{array}{cccc} +a^3 & +b^2a & +c^2a & +d^2a \\ +2a^2b & +2b^3 & +2b^2c & +2d^2b \\ +3a^2c & +3b^2c & +3c^3 & +3d^2c \\ +4a^2d & +4b^2d & +4c^2d & +4d^3 \end{array} < \begin{array}{cccc} +a^3 & +3b^2a & +3c^2a & +3d^2a \\ +3a^2b & +b^3 & +3b^2c & +3d^2b \\ +3a^2c & +3b^2c & +c^3 & +3d^2c \\ +3a^2d & +3b^2d & +3c^2d & +d^3 \\ +6abc & +6bcd & +6cda & +6dab \end{array}$$

In other words, we need to prove that

$$\begin{array}{cccc} & & & \\ & +b^3 & & \\ & & +2c^3 & \\ +a^2d & +b^2d & +c^2d & +3d^3 \end{array} < \begin{array}{cccc} & +2b^2a & +2c^2a & +2d^2a \\ +a^2b & & +b^2c & +d^2b \\ & & & \\ & & & \\ +6abc & +6bcd & +6cda & +6dab \end{array}$$

This follows since

$$\begin{aligned} 2b^2a &\ge b^3 + c^2d \\ 2c^2a &\ge 2c^3 \\ 2d^2a &\ge 2d^3 \\ a^2b &\ge a^2d \\ b^2c &\ge b^2d \\ d^2b &\ge d^3 \end{aligned}$$

and $6(abc+bcd+cda+dab) > 0$.

Remark. Fedor Petrov provides the following motivational comments for why the existence of this solution is not surprising:

> Better to think about mathematics. You have to bound from above a product $(a+2b+3c+4d)(a^2+b^2+c^2+d^2)$, the coefficients $1, 2, 3, 4$ are increasing and so play on your side, so plausibly $(a+b+c+d)^3$ should majorize this term-wise, you check it and this appears to be true.