

IMO 2020/2

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TWITCH SOLVES ISL

Episode 30

Problem

The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

Video

<https://youtu.be/OxeeSsdEgwI>

Solution

By weighted AM-GM we have

$$a^a b^b c^c d^d \leq \sum_{\text{cyc}} \frac{a}{a+b+c+d} \cdot a = a^2 + b^2 + c^2 + d^2.$$

Consequently, it is enough to prove that

$$(a^2 + b^2 + c^2 + d^2)(a + 2b + 3c + 4d) \leq 1 = (a + b + c + d)^3.$$

Expand both sides to get

$$\begin{array}{cccc} +a^3 & +b^2a & +c^2a & +d^2a \\ +2a^2b & +2b^3 & +2b^2c & +2d^2b \\ +3a^2c & +3b^2c & +3c^3 & +3d^2c \\ +4a^2d & +4b^2d & +4c^2d & +4d^3 \end{array} < \begin{array}{cccc} +a^3 & +3b^2a & +3c^2a & +3d^2a \\ +3a^2b & +b^3 & +3b^2c & +3d^2b \\ +3a^2c & +3b^2c & +c^3 & +3d^2c \\ +3a^2d & +3b^2d & +3c^2d & +d^3 \\ +6abc & +6bcd & +6cda & +6dab \end{array}$$

In other words, we need to prove that

$$\begin{array}{cccc} & & & +2b^2a & +2c^2a & +2d^2a \\ & & & +a^2b & +b^2c & +d^2b \\ +b^3 & & & & & & < \\ & +2c^3 & & & & & \\ +a^2d & +b^2d & +c^2d & +3d^3 & +6abc & +6bcd & +6cda & +6dab \end{array}$$

This follows since

$$\begin{aligned} 2b^2a &\geq b^3 + c^2d \\ 2c^2a &\geq 2c^3 \\ 2d^2a &\geq 2d^3 \\ a^2b &\geq a^2d \\ b^2c &\geq b^2d \\ d^2b &\geq d^3 \end{aligned}$$

and $6(abc + bcd + cda + dab) > 0$.