IMO 2020/1

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Twitch Solves ISL

Episode 30

Problem

Consider the convex quadrilateral $ABCD$.. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

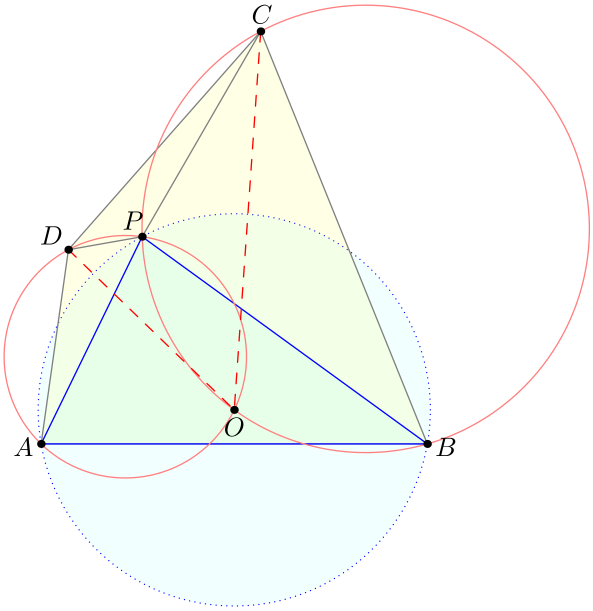
Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Video

https://youtu.be/OxeeSsdEgwI

Solution

Let O denote the circumcenter of $\triangle PAB$. We claim it is the desired concurrency point.



Indeed, O obviously lies on the perpendicular bisector. Now

$$\begin{aligned} \measuredangle BCP &= \measuredangle CBP + \measuredangle BPC \\ &= 2\measuredangle BAP = \measuredangle BOP \end{aligned}$$

it follows $BOPC$ are cyclic. And since $OP = OB$, it follows that O is on the bisector of $\angle PCB$, as needed.

Remark. The angle equality is only used insomuch $\angle BAP$ is the average of $\angle CBP$ and $\angle BPC$, i.e. only $\frac{1+3}{2} = 2$ matters.