

# SMO 2020/3

Evan Chen

TWITCH SOLVES ISL

Episode 29

## Problem

Let  $\triangle ABC$  be an acute scalene triangle with incenter  $I$  and incircle  $\omega$ . Two points  $X$  and  $Y$  are chosen on minor arcs  $AB$  and  $AC$ , respectively, of the circumcircle of triangle  $\triangle ABC$  such that  $XY$  is tangent to  $\omega$  at  $P$  and  $\overline{XY} \perp \overline{AI}$ . Let  $\omega$  be tangent to sides  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Denote the intersection of lines  $XF$  and  $YE$  as  $T$ .

Prove that if the circumcircles of triangles  $\triangle TEF$  and  $\triangle ABC$  are tangent at some point  $Q$ , then lines  $PQ$ ,  $XE$ , and  $YF$  are concurrent.

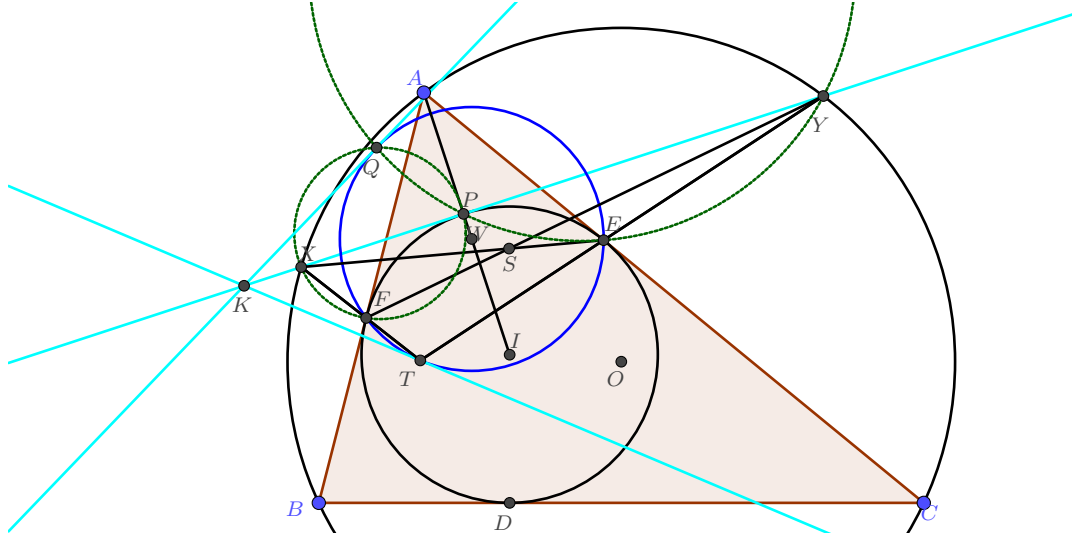
## Video

<https://youtu.be/rScZrRpRkR0>

### Solution

We ignore the condition that  $(TEF)$  and  $(ABC)$  are tangent until the very end.

Instead, we define  $Q$  as the intersection of the circles  $(TEF)$ ,  $(XPF)$ ,  $(YPE)$  — this point exists by Miquel's theorem on  $\triangle TXY$  with  $E \in \overline{TY}$ ,  $F \in \overline{TX}$ ,  $P \in \overline{XY}$ .



**Claim.** Point  $Q$  lies on  $\overline{PS}$ .

*Proof.* It suffices to show  $S$  lies on the radical axis of  $(XPF)$  and  $(YPE)$ . Let  $\overline{ESX}$  meet  $(XPF)$  again at  $Z_1$  and let  $\overline{ESY}$  meet  $(YPE)$  again at  $Z_2$ . Then

$$\angle SZ_1F = \angle XZ_1F = \angle XPF = \angle YPE = \angle YZ_2E = \angle SZ_2E$$

which implies  $Z_1, Z_2, E, F$  are cyclic.

On the other hand, since  $\overline{EF} \parallel \overline{XY}$  (both are perpendicular to  $\overline{AI}$ ), it follows that  $SX : SY = SE : SF$ . So we are able to get  $SZ_1 \cdot SX = SZ_2 \cdot SY$  as needed.  $\square$

Let  $W$  denote the circumcenter of  $TFEQ$  which lies on  $\overline{API}$ .

**Claim.** Line  $XY$  passes through the pole  $K$  of line  $QT$  with respect to  $(TFEQ)$ .

*Proof.* We have  $\angle QPW = \angle QPX + 90^\circ = \angle QFX + 90^\circ = \angle QFT + 90^\circ$  which can be checked to equal  $\angle QTW$  is computable. Hence  $QPWT$  cyclic. This circle also passes through the pole of  $\overline{QT}$ , and since  $\overline{WP} \perp \overline{XY}$ , the conclusion follows.  $\square$

So if the two circles are tangent, then the tangency point must coincide with the  $Q$  we found. And  $Q$  already lies on  $PS$ .