SMO 2020/3

Evan Chen

Twitch Solves ISL

Episode 29

Problem

Let $\triangle ABC$ be an acute scalene triangle with incenter I and incircle ω. Two points X and Y are chosen on minor arcs AB and AC, respectively, of the circumcircle of triangle $\triangle ABC$ such that XY is tangent to ω at P and $\overline{XY} \perp \overline{AI}$. Let ω be tangent to sides AC and AB at E and F, respectively. Denote the intersection of lines XF and YE as T.

Prove that if the circumcircles of triangles $\triangle TEF$ and $\triangle ABC$ are tangent at some point Q, then lines PQ, XE, and YF are concurrent.

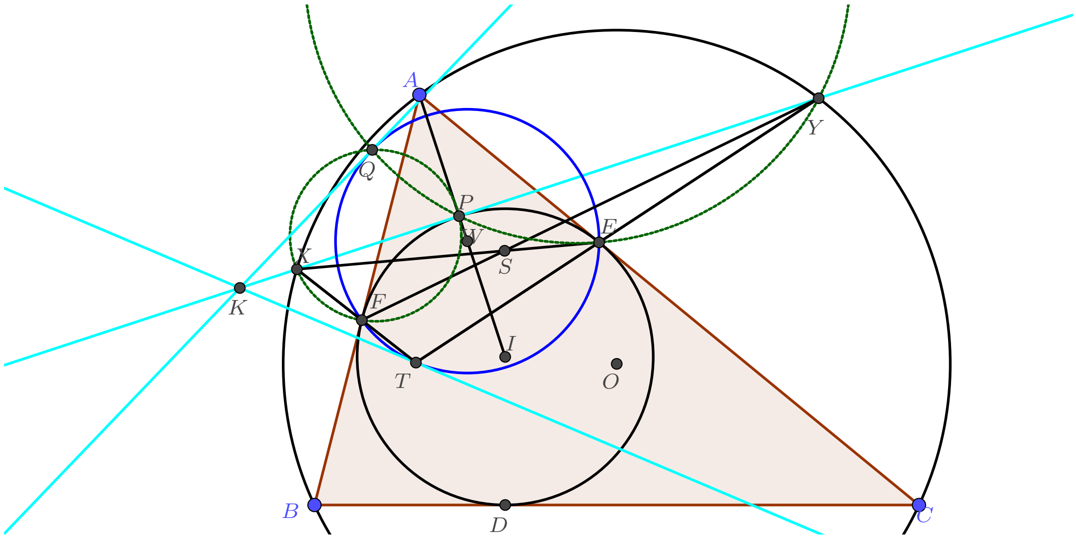
Video

https://youtu.be/rScZrRpRkR0

Solution

We ignore the condition that (TEF) and (ABC) are tangent until the very end.

Instead, we define Q as the intersection of the circles (TEF), (XPF), (YPE) — this point exists by Miquel's theorem on $\triangle TXY$ with $E \in \overline{TY}$, $F \in \overline{TX}$, $P \in \overline{XY}$.



Claim. Point Q lies on $\overline{PS}$.

Proof. It suffices to show S lies on the radical axis of (XPF) and (YPE). Let $\overline{ESX}$ meet (XPF) again at Z_1 and let $\overline{ESY}$ meet (YPE) again at Z_2. Then

$$\measuredangle SZ_1F = \measuredangle XZ_1F = \measuredangle XPF = \measuredangle YPE = \measuredangle YZ_2E = \measuredangle SZ_2E$$

which implies Z_1, Z_2, E, F are cyclic.

On the other hand, since $\overline{EF} \parallel \overline{XY}$ (both are perpendicular to $\overline{AI}$), it follows that $SX : SY = SE : SF$. So we are able to get $SZ_1 \cdot SX = SZ_2 \cdot SY$ as needed. $\square$

Let W denote the circumcenter of $TFEQ$ which lies on $\overline{API}$.

Claim. Line XY passes through the pole K of line QT with respect to $(TFEQ)$.

Proof. We have $\measuredangle QPW = \measuredangle QPX + 90^\circ = \measuredangle QFX + 90^\circ = \measuredangle QFT + 90^\circ$ which can be checked to equal $\measuredangle QTW$ is computable. Hence $QPWT$ cyclic. This circle also passes through the pole of $\overline{QT}$, and since $\overline{WP} \perp \overline{XY}$, the conclusion follows. $\square$

So if the two circles are tangent, then the tangency point must coincide with the Q we found. And Q already lies on PS.