IMO 1999/4

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Twitch Solves ISL

Episode 29

Problem

Find all pairs of positive integers (x, p) such that p is a prime and x^{p-1} is a divisor of $(p-1)^x + 1$.

Video

https://youtu.be/HybwwvQQPHA

External Link

https://aops.com/community/p131811

Solution

If $p = 2$ then $x \in \{1, 2\}$, and if $p = 3$ then $x \in \{1, 3\}$, since this is IMO 1990/3. Also, $x = 1$ gives a solution for any prime p. We show that there are no other solutions.

Assume $x > 1$ and let q be smallest prime divisor of x. We have $q > 2$ since $(p-1)^x + 1$ is odd. Then

$$(p-1)^x \equiv -1 \pmod q \implies (p-1)^{2x} \equiv 1 \pmod q$$

so the order of $p-1 \bmod q$ is even and divides $\gcd(q-1, 2x) \le 2$. This means that

$$p - 1 \equiv -1 \pmod q \implies p = q.$$

In other words $p \mid x$ and we get $x^{p-1} \mid (p-1)^x + 1$. By exponent lifting lemma, we now have

$$0 < (p-1)\nu_p(x) \le 1 + \nu_p(x).$$

This forces $p = 3$, which we already addressed.