China TST 2019/4/1

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Twitch Solves ISL

Episode 29

Problem

Let $ABCD$ be a cyclic quadrilateral inscribed in a circle with center O. Points M and N are the midpoints of $\overline{BC}$ and $\overline{CD}$, and points E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Lines EN meet FM at P. Let S denote the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

Video

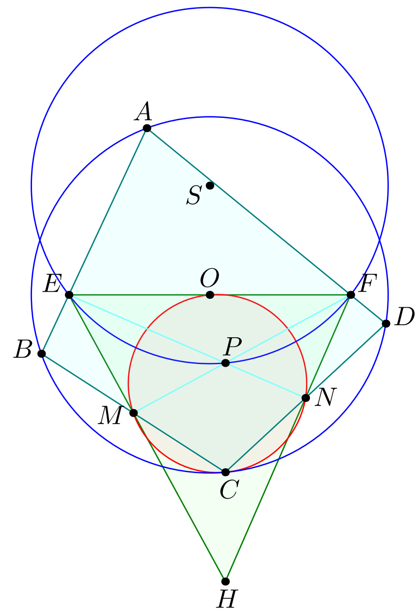
https://youtu.be/OxeeSsdEgwI

External Link

https://aops.com/community/p12139797

Solution

We let H denote the orthocenter of $\triangle PEF$.



Let ω denote the circle with diameter $\overline{OC}$, passing through M and N.

Claim. The circle ω is the nine-point circle of $\triangle PEF$ (or $\triangle HEF$ if you prefer).

Proof. We observe a few facts:

- Clearly ω has radius half that of (O). Since $SP = OC$, the circles (S) and (S) are congruent, hence the radius of ω is half that of (PEF) too.
- Point O is the midpoint of $\overline{EF}$,
- The antipode of O — namely C — is known to lie on the P-altitude (because $\overline{SO} \perp \overline{CP}$ and $\overline{SO} \parallel \overline{EF}$). $\square$

Claim. $\overline{AC}$ bisects $\angle BAD$.

Proof. We have $OM = ON$, so $BC = CD$. $\square$

Claim. We have $\overline{CA} \parallel \overline{OP}$.

Proof. From $ABCD$ is cyclic, we can compute

$$\measuredangle EAF = \measuredangle BAD = \measuredangle BCD = \measuredangle MCN = \measuredangle MON = 2\measuredangle FHE$$

so A lies on the circle through E, F, and the circumcenter of $\triangle HEF$. Denote this circumcenter by W. As $WE = WF$, so this implies $\overline{AW}$ bisects $\angle EAF$, and hence AWC are collinear. Since $OWCP$ is a parallelogram, this completes the proof. $\square$

This completes the solution.