SMO 2020/1

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Twitch Solves ISL

Episode 28

Problem

The sequence of positive integers $a_0, a_1, a_2, \dots$ is recursively defined such that a_0 is not a power of 2, and for all nonnegative integers n:

(i) if a_n is even, then a_{n+1} is the largest odd factor of a_n and

(ii) if a_n is odd, then $a_{n+1} = a_n + p^2$ where p is the smallest prime factor of a_n.

Prove that there exists some positive integer M such that $a_{m+2} = a_m$ for all $m \geq M$.

Video

https://youtu.be/yGeCeBOMyA4

Solution

The sequence alternates even and odd, so WLOG $a_1, a_3, \ldots$ are the odd numbers.

Note that the sequence of smallest prime divisors of a_1, a_3, $\ldots$is a weakly decreasing sequence and so it is eventually stable — so we may as well assume that $p \mid a_1, a_3, \ldots$ and no term of the sequence isn't divisible by any smaller prime.

We consider two cases:

- If eventually the sequence contains p itself, then in fact $p+1$ is a power of 2, and so if $a_{2k+1} = p$ then $a_{2k+3} = a_{2k+5} = \cdots = p$.
- Otherwise, note that given $a_{2k+1} \neq p$ odd, we have $a_{2k+3} \leq \frac{1}{2}(a_{2k+1} + p^2) \leq a_{2k+1}$, so the sequence of odd numbers is weakly decreasing. Hence it stabilizes.