SJMO 2020/3

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Twitch Solves ISL

Episode 28

Problem

Let O and Ω denote the circumcenter and circumcircle, respectively, of scalene triangle $\triangle ABC$. Furthermore, let M be the midpoint of side BC. The tangent to Ω at A intersects BC and OM at points X and Y, respectively. If the circumcircle of triangle $\triangle OXY$ intersects Ω at two distinct points P and Q, prove that line PQ bisects $\overline{AM}$.

Video

https://youtu.be/bI_9MjmoRgE

Solution

In other words, we want the midpoint of $\overline{AM}$ to lie on the radical axis of the two circles.

However, we have

$$\begin{aligned} \operatorname{Pow}(M, \Omega) &= -\frac{1}{4}BC^2 \\ \operatorname{Pow}(A, \Omega) &= 0 \\ \operatorname{Pow}(M, OXY) &= MO \cdot MY \\ \operatorname{Pow}(A, OXY) &= -AX \cdot AY. \end{aligned}$$

Since the function $\operatorname{Pow}(\bullet, \Omega) - \operatorname{Pow}(\bullet, OXY)$ is linear, it thus suffices to show that

$$AX \cdot AY = \frac{1}{4}BC^2 + MO \cdot MY$$

where the lengths are signed.

Let $YA = h$, $AO = OB = R$, so $YO = \sqrt{R^2 + h^2}$. Let $YM = kh$, $YX = k\sqrt{r^2 + h^2}$, and $XM = kh$. Note that $\frac{1}{4}BC^2 = BM^2 = R^2 - OM^2$, and overall we find our desired relation reads

$$\left(k\sqrt{R^2 + h^2} - h\right) \cdot h = \left(R^2 - \left(kh - \sqrt{R^2 + h^2}\right)^2\right) + \left(k \cdot h - \sqrt{R^2 + h^2}\right) \cdot k \cdot h$$

which is easily verified.