KoMaL A774

Evan Chen

Twitch Solves ISL

Episode 28

Problem

Let O be the circumcenter of triangle ABC, and D be an arbitrary point on the circumcircle of ABC. Let points X, Y and Z be the orthogonal projections of point D onto lines OA, OB and OC, respectively. Prove that the incenter of triangle XYZ is on the Simson line of triangle ABC corresponding to point D.

Video

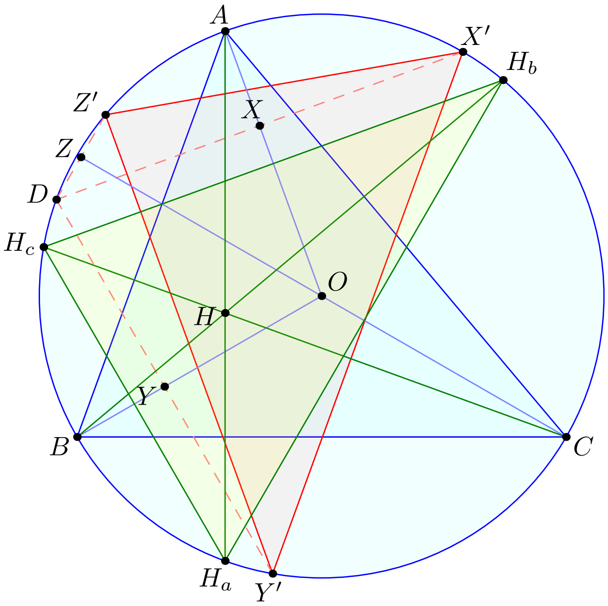
https://youtu.be/g-nMGCXyCSM

Solution

We let X' be the reflection of X across $\overline{OA}$, which lies on (O), and define Y', Z' similarly. We denote by J the incenter of $\triangle X'Y'Z'$.

Claim. $\triangle X'Y'Z'$ is oppositely similar to the triangle whose vertices coincide with the reflections of the orthocenter of ABC over the sides of ABC.

Proof. Immediate by angle chasing. $\square$



We now use complex numbers. We know $x' = a^2/d$, etc. Also, the special triangle we mentioned previously has vertices $-bc/a$, $-ca/b$, $-ab/c$ and orthocenter $a+b+c$. The map between them is

$$z \mapsto -\frac{abc}{d}\overline{z}.$$

So we conclude the coordinates for J

$$-\frac{ab+bc+ca}{d}.$$

We now calculate

$$\begin{aligned}
&\det \begin{bmatrix} -\frac{ab+bc+ca}{d} & -\frac{d(a+b+c)}{abc} & 1 \\ \frac{d(a+b)-ab}{d} & \frac{a+b-d}{ab} & 1 \\ \frac{d(a+c)-ac}{d} & \frac{a+c-d}{ac} & 1 \end{bmatrix} \\
&= \frac{1}{abcd} \det \begin{bmatrix} -(ab+bc+ca) & -d(a+b+c) & 1 \\ d(a+b)-ab & c(a+b-d) & 1 \\ d(a+c)-ac & b(a+c-d) & 1 \end{bmatrix} \\
&= \frac{1}{abcd} \det \begin{bmatrix} -(ab+bc+ca) & -d(a+b+c) & 1 \\ (d-a)(b-c) & (a-d)(c-b) & 0 \\ d(a+c)-ac & b(a+c-d) & 1 \end{bmatrix} \\
&= \frac{(a-d)(b-c)}{abcd} \det \begin{bmatrix} -(ab+bc+ca) & -d(a+b+c) & 1 \\ 1 & 1 & 0 \\ d(a+c)-ac & b(a+c-d) & 1 \end{bmatrix} \\
&= \text{const}\big[b(a+c-d) + d(a+b+c) \\
&\qquad - (ab+bc+ca) - (d(a+c)-ac)\big] = 0.
\end{aligned}$$

This proves it lies on Simson line.

Remark (Darij Grinberg)**.** Here is a synthetic ending that avoids the complex numbers calculation:

Show that the line H_aX' is parallel to the Simson line of D (this should be angle chasing). The same must then be true for H_bY' and H_cZ' by analogy. Since all these six points lie on the same circle, this means that X', Y', Z' are the reflections of H_a, H_b, H_c in the same line, which is the perpendicular from O to the Simson line of D. Thus, the incenter of $X'Y'Z'$ is the reflection of the incenter of $H_aH_bH_c$ in the same line. But the latter incenter is known to be H. Now use the homothety with factor $1/2$ from D and recall that the Simson line of D bisects DH.