IMO 1981/3

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Twitch Solves ISL

Episode 28

Problem

Determine the maximum possible value of $m^2 + n^2$ where m and n are integers in $\{1, 2, \dots, 1981\}$ satisfying $(n^2 - mn - m^2)^2 = 1$.

Video

https://youtu.be/U8xUT782kts

External Link

https://aops.com/community/p366642

Solution

We start by just applying the quadratic formula. Write

$$n^2 - mn - m^2 = \pm 1 \qquad n = \frac{m \pm \sqrt{5m^2 \pm 4}}{2}$$

Hence for any given m there is a valid n iff $5m^2 \pm 4$ is a square.

It turns out that:

Claim. A positive integer m is a Fibonacci number if and only if at least one of $5m^2 - 4$ and $5m^2 + 4$ is a perfect square.

You're welcome.

A quick calculation then gives $(m, n) = (987, 1597)$ so the optimal value is $987^2 + 1597^2$.

Remark. If you don't know this claim about Fibonacci numbers, you would end up finding it by solving the resulting Pell equation. So while it's funny that it turns out $5m^2 \pm 4 = t^2$ gives exactly the Fibonacci numbers for m, the solution doesn't really depend on this and would work with other numbers too.