IMO 1981/3

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Twitch Solves ISL

Episode 28

Problem

Determine the maximum value of $m^3 + n^3$ where m and n are integers in the range $1, 2, \dots, 1981$ satisfying $(n^2 - mn - m^2)^2 = 1$.

Video

https://youtu.be/U8xUT782kts

Solution

Write

$$n^2 - mn - m^2 = \pm 1 \qquad n = \frac{m \pm \sqrt{5m^2 \pm 4}}{2}$$

Hence for any given m there is a valid n iff $5m^2 \pm 4$ is a square.

Well, it turns out this is equivalent to m being a Fibonacci number. You're welcome.

A quick calculation then gives $(m, n) = (987, 1597)$ so the optimal value is $987^3 + 1597^3$.