CAMO 2019/2

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Twitch Solves ISL

Episode 28

Problem

Let k be a positive integer, $p > 3$ a prime, and n an integer with $0 \le n \le p^{k-1}$. Prove that

$$\binom{p^k}{pn} \equiv \binom{p^{k-1}}{n} \pmod{p^{2k+1}}.$$

Video

https://youtu.be/5hMZamIYjD8

Solution

Main claim:

Claim. For$c \ge 0$, $k \ge 1$, $p > 3$ prime, we hav

$$\begin{aligned} &\left(p^k - (cp+1)\right)\left(p^k - (cp+2)\right)\dots\left(p^k - (cp+(p-1))\right) \\ &\equiv (cp+1)(cp+2)\dots(cp+(p-1)) \pmod{p^{k+2}} \end{aligned}$$

Proof. Expanding as a "polynomial in p^k" we find it's sufficient to show

$$\frac{1}{cp+1} + \frac{1}{cp+2} + \dots + \frac{1}{cp+(p-1)} \equiv 0 \pmod{p^2}$$

which is Wolstenholme's theorem, proved by simply pairing the opposite fractions together. □

Now it is enough to note that

$$\binom{p^k}{pn} = p^{k-1}\binom{p^{k-1}}{n}\prod_{c=0}^{n-1}\prod_{i=1}^{p-1}\frac{p^k-(cp+i)}{cp+i}.$$

This gives the result.