SMO 2020/5

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Twitch Solves ISL

Episode 27

Problem

In triangle ABC, let E and F be points on sides AC and AB, respectively, such that $BFEC$ is cyclic. Let lines BE and CF intersect at point P, and M and N be the midpoints of $\overline{BF}$ and $\overline{CE}$, respectively. If U is the foot of the perpendicular from P to BC, and the circumcircles of $\triangle BMU$ and $\triangle CNU$ intersect at second point V different from U, prove that A, P, and V are collinear.

Video

https://youtu.be/XlXnHk9AtAE

External Link

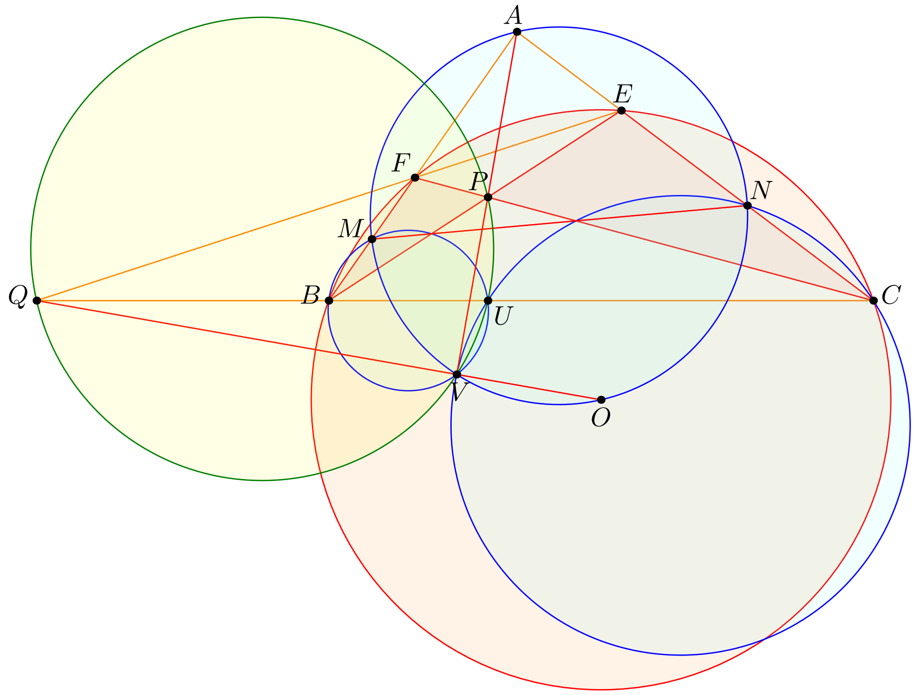
https://aops.com/community/p17350803

Solution

We show two approaches.

Classical approach, by Evan. We redefine V to be the Miquel point of self-intersecting cyclic quadrilateral $BFCE$. So for example, this automatically implies V is the perpendicular intersection of lines $\overline{APV}$ and $\overline{OQ}$, and we wish to show it lies on (BMU) and (CNU).

We also let $Q = \overline{EF} \cap \overline{BC}$.



Claim. The point V lies on $(AMON)$.

Proof. Follows from $\angle AVO = 90^\circ$. This follows from the fact that the spiral similarity at V which maps $\overline{BF}$ to $\overline{EC}$ also maps M to N. Hence V is the Miquel point of $BMNE$, ergo the intersection of (BMX) and (AMN). □

Therefore, all that remains is to show that U lies on $(BMXV)$ and $(CNYV)$, as defined.

Claim. We have $PUVQ$ is cyclic.

Proof. Follows from $\measuredangle PUQ = \measuredangle PVQ = 90^\circ$. □

Finally,

$$\measuredangle BUV = \measuredangle QUV = \measuredangle QPV = \measuredangle QPA = \measuredangle AOQ = \measuredangle AMV = \measuredangle BMV$$

as needed.

Coaxiality approach, via MarkBCC. See https://aops.com/community/p17361875 where the forgotten coaxiality lemma is used to show $PQUV$ are concyclic.