IMO 1990/6

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Twitch Solves ISL

Episode 27

Problem

Prove that there exists a convex equiangular 1990-gon whose side lengths are 1^2, 2^2, 3^2, $\cdots$, 1990^2 in some order.

Video

https://youtu.be/rTb16n8eAU8

Solution

Throughout this solution, ω denotes a primitive 995th root of unity.

We first commit to placing 1^2 and 2^2 on opposite sides, 3^2 and 4^2 on opposite sides, etc. Since $2^2 - 1^2 = 3$, $4^2 - 3^2 = 7$, $6^2 - 5^2 = 11$, etc., this means the desired conclusion is equivalent to

$$0 = \sum_{n=0}^{994} c_n \omega^n$$

being true for some permutation $(c_0, \dots, c_{994})$ of $(3, 7, 11, \dots, 3981)$.

Define $z = 3\omega^0 + 7\omega^{199} + 11\omega^{398} + 15\omega^{597} + 19\omega^{796}$. Then notice that

$$\begin{aligned} z &= 3\omega^0 + 7\omega^{199} + 11\omega^{398} + 15\omega^{597} + 19\omega^{796} \\ \omega^5 z &= 23\omega^5 + 27\omega^{204} + 31\omega^{403} + 35\omega^{602} + 39\omega^{801} \\ \omega^{10} z &= 43\omega^{10} + 47\omega^{209} + 51\omega^{408} + 55\omega^{607} + 59\omega^{806} \\ &\vdots \end{aligned}$$

and so summing yields the desired conclusion, as the left-hand side becomes

$$(1 + \omega^5 + \omega^{10} + \dots + \omega^{990})z = 0$$

and the right-hand side is the desired expression.