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Twitch Solves ISL

Episode 27

Problem

Let ABC be a scalene triangle whose incircle touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. Let L, M, N denote the reflections of D across $\overline{EF}$, E across $\overline{FD}$, F across $\overline{DE}$. Let $P = \overline{AL} \cap \overline{BC}$, $Q = \overline{BM} \cap \overline{CA}$, $R = \overline{CN} \cap \overline{AB}$. Prove that P, Q, R are collinear.

Video

https://youtu.be/X3Nw-GWac9k

External Link

https://aops.com/community/p2628489

Solution

We use complex numbers with $\triangle DEF$ as the unit circle. To compute P, we know that

$$p + d^2\overline{p} = 2d$$

and

$$\begin{aligned} 0 &= \det \begin{bmatrix} p & \overline{p} & 1 \\ \frac{2ef}{e+f} & \frac{2}{e+f} & 1 \\ e+f-\frac{ef}{d} & \frac{1}{e}+\frac{1}{f}-\frac{d}{ef} & 1 \end{bmatrix} \\ \implies 0 &= \det \begin{bmatrix} p & \overline{p} & 1 \\ 2ef & 2 & e+f \\ ef[d(e+f)-ef] & d(e+f-d) & def \end{bmatrix} \\ &= d\left[d(e+f)-e^2-f^2\right]p \\ &\qquad - ef\left[ef(e+f)-d(e^2+f^2)\right]\overline{p} + 2ef\left[ef-d^2\right]. \end{aligned}$$

Eliminating $\overline{p}$ then lets one explicitly compute p:

$$\begin{aligned} p &= \frac{\left[(d^2-ef)\cdot 2ef\right]\cdot d^2 + ef\left[ef(e+f)-d(e^2+f^2)\right]\cdot 2d}{d\left[d(e+f)-e^2-f^2\right]\cdot d^2 + ef\left[ef(e+f)-d(e^2+f^2)\right]\cdot 1} \\ &= 2def \cdot \frac{d(d^2-ef)+ef(e+f)-d(e^2+f^2)}{d^3\left[d(e+f)-e^2-f^2\right]+ef\left[ef(e+f)-d(e^2+f^2)\right]} \\ &= 2def \cdot \frac{d^3-(e^2+ef+f^2)d+ef(e+f)}{(e+f)d^4-(e^2+f^2)d^3-ef(e^2+f^2)d+e^2f^2(e+f)} \end{aligned}$$

Now

$$p - \frac{2df}{d+f} = 2df\left[\frac{e(d^3-(e^2+ef+f^2)d+ef(e+f))}{(e+f)d^4-(e^2+f^2)d^3-ef(e^2+f^2)d+e^2f^2(e+f)} - \frac{1}{d+f}\right]$$

The bulk of the work is expanding the numerator of the resulting fraction which equals

$$\begin{aligned} & e(d+f)(d^3-(e^2+ef+f^2)d+ef(e+f)) \\ &\qquad - \left[(e+f)d^4-(e^2+f^2)d^3-ef(e^2+f^2)d+e^2f^2(e+f)\right] \\ &= -f\cdot d^4 + (e^2+ef+f^2)d^3 - e(e^2+ef+f^2)d^2 + e^3fd \\ &= d\left[f(e^3-d^3)+d(e^2+ef+f^2)(d-e)\right] \\ &= d\left[(d-e)(d(e^2+ef+f^2)-f(d^2+de+e^2))\right] \\ &= d(d-e)\left(de^2+df^2-fd^2-fe^2\right) \\ &= d(d-e)(d-f)(e^2-df). \end{aligned}$$

Thus, we have that

$$p - b = \frac{2df}{d+f}\cdot\frac{d(d-e)(d-f)(e^2-df)}{(e+f)d^4-(e^2+f^2)d^3-ef(e^2+f^2)d+e^2f^2(e+f)}$$

Now, it follows that

$$\frac{p-b}{p-c} = \frac{f(d+e)(e^2-df)}{e(d+f)(f^2-de)}.$$

Multiplying cyclically gives $\frac{p-b}{p-c}\frac{q-c}{q-a}\frac{r-a}{r-b} = 1$. Therefore, by Menelaus theorem, the problem is solved.

Remark. Actually, it turns out that P, Q, R lie on the Euler line of triangle DEF, and there is a factor of $d+e+f$ hidden in the numerator of P. So a simpler approach would be to factor this out from P, and then notice the resulting quotient is real.