SJMO 2020/2

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TWITCH SOLVES ISL

Episode 26

Problem

Anthony writes the $(n+1)^2$ distinct positive integer divisors of 10^n, each once, on a whiteboard. On a move, he may choose any two distinct numbers a and b on the board, erase them both, and write $\gcd(a,b)$ twice. Anthony keeps making moves until all of the numbers on the board are the same. Find the minimum possible number of moves Anthony could have made.

Video

Solution

The answer is $n + n^2$. This is achieved by doing the following algorithm:

- For $i = 1, \ldots, n$ erase 2^i and 5^i and replace both with 1.

- For any of the other n^2 other numbers x on the board larger than 1 after this, erase x and 1 and replace both with 1.

To see this is optimal, define the *score* of a number as 0 if the number is one, 1 if the number is a power of 2 or a power of 5 other than 1, and 2 otherwise.

Claim. The total score of all numbers decreases by at most 2 each step.

Proof. Obvious. $\qquad\square$

Since the total score to start is $2(n^2 + n)$, we are done.