

SJMO 2020/2

Evan Chen

TWITCH SOLVES ISL

Episode 26

Problem

Anthony writes the $(n + 1)^2$ distinct positive integer divisors of 10^n , each once, on a whiteboard. On a move, he may choose any two distinct numbers a and b on the board, erase them both, and write $\gcd(a, b)$ twice. Anthony keeps making moves until all of the numbers on the board are the same. Find the minimum possible number of moves Anthony could have made.

Video

<https://youtu.be/uLAvYA0CpSo>

Solution

The answer is $n + n^2$. This is achieved by doing the following algorithm:

- For $i = 1, \dots, n$ erase 2^i and 5^i and replace both with 1.
- For any of the other n^2 other numbers x on the board larger than 1 after this, erase x and 1 and replace both with 1.

To see this is optimal, define the *score* of a number as 0 if the number is one, 1 if the number is a power of 2 or a power of 5 other than 1, and 2 otherwise.

Claim. The total score of all numbers decreases by at most 2 each step.

Proof. Obvious. □

Since the total score to start is $2(n^2 + n)$, we are done.