# Brazil 2017/6 

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Twitch Solves ISL

Episode 26

## Problem

Let $a$ be a positive integer and $p$ a prime divisor of $a^{3}-3 a+1$, with $p \neq 3$. Prove that $p$ is of the form $9 k+1$ or $9 k-1$, where $k$ is integer.

## Video

https://youtu.be/A3AQVRvVk3g

## External Link

https://aops.com/community/p9495218

## Solution

Write $a=x+\frac{1}{x}$ for some $x \in \mathbb{F}_{p^{2}}$.
Claim. The element $x$ has order 9 .
Proof. Because

$$
\begin{aligned}
0 & =\left(x+\frac{1}{x}\right)^{3}-3\left(x+\frac{1}{x}\right)+1 \\
& =x^{3}+x^{-3}+1=\frac{x^{6}+x^{3}+1}{x^{3}}
\end{aligned}
$$

This implies $x^{9}=1$, so $x$ has order dividing 9 . However, $x^{3} \neq 1$ since $p>3$. Therefore, $x$ has order exactly 9 .

Thus $9 \mid p^{2}-1$ so we're done.
Remark. Useful example for debugging: $a=7$ gives $p=17$. (Sometimes people think $9 \mid p-1$ should follow.)

