

SIME 2020/13

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TWITCH SOLVES ISL

Episode 25

Problem

In acute triangle $\triangle ABC$, $AB = 20$ and $AC = 21$. Let the feet of the perpendiculars from A to the angle bisectors of $\angle ACB$ and $\angle ABC$ be X and Y , respectively. Let M be the midpoint of \overline{XY} . Suppose P is the point on side BC such that MP is parallel to the angle bisector of $\angle BAC$. If given that $BP = 11$, find the length of BC .

Video

<https://youtu.be/hQ4lFgub7tU>

Solution

The answer is $BC = 205/9$.

Introduce the intouch triangle DEF .

Claim. Point M is the midpoint of \overline{AD} .

Proof. By the so-called Iran lemma, we have X on line DF and Y on line EF , and it follows $DXAY$ is a parallelogram. \square

Let K be the foot of angle bisector. Then it follows that $BP = \frac{1}{2}(BD + BK)$. Thus we may write in the usual notation

$$\begin{aligned} 11 = BP &= \frac{(s-b) + \frac{c}{c+b} \cdot a}{2} = \frac{\frac{a-1}{2} + \frac{20}{41}a}{2} \\ \implies \frac{45}{2} &= \frac{81}{82} \cdot a \implies a = \frac{205}{9}. \end{aligned}$$