# SIME 2020/13 

Evan Chen

## Twitch Solves ISL

Episode 25

## Problem

In acute triangle $\triangle A B C, A B=20$ and $A C=21$. Let the feet of the perpendiculars from $A$ to the angle bisectors of $\angle A C B$ and $\angle A B C$ be $X$ and $Y$, respectively. Let $M$ be the midpoint of $\overline{X Y}$. Suppose $P$ is the point on side $B C$ such that $M P$ is parallel to the angle bisector of $\angle B A C$. Given that $B P=11$, find the length of $B C$.

## Video

https://youtu.be/hQ4lFgub7tU

## Solution

The answer is $B C=205 / 9$. Introduce the intouch triangle $D E F$.


Claim. Point $M$ is the midpoint of $\overline{A D}$.
Proof. By EGMO Lemma 1.45, we have $X$ on line $D F$. Since $\overline{D F X} \perp \overline{B I}$ and $\overline{B I} \perp \overline{A Y}$, we have $\overline{D X} \| \overline{A Y}$. Similarly, $\overline{D Y} \| \overline{A X}$, and it follows $D X A Y$ is a parallelogram.

Let $K$ be the foot of angle bisector. Then it follows that $B P=\frac{1}{2}(B D+B K)$. Thus we may write in the usual notation

$$
\begin{aligned}
11 & =B P=\frac{(s-b)+\frac{c}{c+b} \cdot a}{2}=\frac{\frac{a-1}{2}+\frac{20}{41} a}{2} \\
\Longrightarrow \frac{45}{2} & =\frac{81}{82} \cdot a \Longrightarrow a=\frac{205}{9} .
\end{aligned}
$$

