SIME 2020/13

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Twitch Solves ISL

Episode 25

Problem

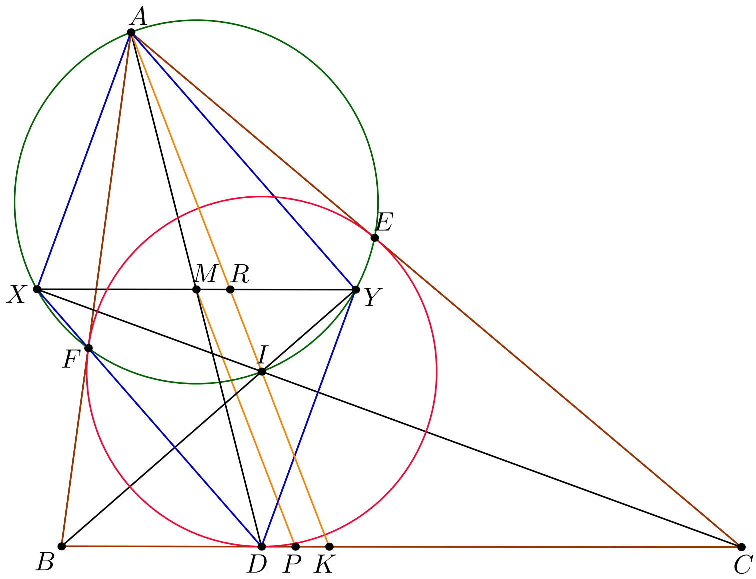
In acute triangle $\triangle ABC$, $AB = 20$ and $AC = 21$. Let the feet of the perpendiculars from A to the angle bisectors of $\angle ACB$ and $\angle ABC$ be X and Y, respectively. Let M be the midpoint of $\overline{XY}$. Suppose P is the point on side BC such that MP is parallel to the angle bisector of $\angle BAC$. Given that $BP = 11$, find the length of BC.

Video

https://youtu.be/hQ4lFgub7tU

Solution

The answer is $BC = 205/9$. Introduce the intouch triangle DEF.



Claim. Point M is the midpoint of $\overline{AD}$.

Proof. By EGMO Lemma 1.45, we have X on line DF. Since $\overline{DFX} \perp \overline{BI}$ and $\overline{BI} \perp \overline{AY}$, we have $\overline{DX} \parallel \overline{AY}$. Similarly, $\overline{DY} \parallel \overline{AX}$, and it follows $DXAY$ is a parallelogram. $\square$

Let K be the foot of angle bisector. Then it follows that $BP = \frac{1}{2}(BD + BK)$. Thus we may write in the usual notation

$$\begin{aligned} 11 = BP &= \frac{(s-b) + \frac{c}{c+b} \cdot a}{2} = \frac{\frac{a-1}{2} + \frac{20}{41}a}{2} \\ \implies \frac{45}{2} &= \frac{81}{82} \cdot a \implies a = \frac{205}{9}. \end{aligned}$$