MODS 2019/2

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Twitch Solves ISL

Episode 25

Problem

Find all positive integers b such that

$$1 + b + b^2 + \cdots + b^e$$

is a triangular number for every positive integer e.

Video

https://youtu.be/EE3z66dcd4s

Solution

The answer is $b = 9$ only, which works since

$$1 + 9 + \cdots + 9^e = \frac{1}{2} \frac{3^{e+1} + 1}{2} \frac{3^{e+1} - 1}{2}.$$

To prove $b = 9$ is the only one possible, note that for $e = 2$ and $e = 4$ we find that

$$8(b+1) + 1 \qquad \text{and} \qquad 8(b^3 + b^2 + b + 1) + 1$$

are perfect squares. Hence the product A defined by

$$\begin{aligned} A &\stackrel{\text{def}}{=} 4096\,[8(b+1)+1]\left[8(b^3+b^2+b+1)+1\right] \\ &= 4096(8b+9)(8b^3+8b^2+8b+9) \\ &= 4096(64b^4+136b^3+136b^2+144b+81) \\ &= 4096 \cdot \left[\left(8b^2 + \frac{17}{2}b + \frac{255}{64}\right)^2 + \frac{4881b}{64} + \frac{266751}{4096}\right] \\ &= \left(512b^2+544b+255\right)^2 + (312384b + 266751) \end{aligned}$$

is a perfect square divisible by 4096.

However, $312384b + 266751 < 33(1024b^2 + 1088b + 533)$ holds for all $b \geq 10$. So when $b \geq 10$ we have $(512b^2+544b+255)^2 < A < (512b^2+544b+288)^2$. Since A was supposed to be a perfect square divisible by 4096, one would have to have $A = (512b^2+544b+256)^2$, so

$$\begin{aligned} (512b^2+544b+256)^2 &= \left(512b^2+544b+255\right)^2 + (312384b+266751) \\ \iff 312384b + 266751 &= 1024b^2 + 1088b + 511 \\ \iff 0 &= 1024b^2 - 311296b + 266240 \\ \iff 0 &= b^2 - 304b + 260 \end{aligned}$$

But the discriminant $304^2 - 4 \cdot 260$ isn't a square, so this is a contradiction.

Hence, it suffices to exclude $b = 1, 2, \dots, 8$ exhaustively. For $1 + b$ to be a triangular number for $b \leq 8$, one needs $b \in \{2, 5\}$, but $1 + 2 + 2^2 = 7$ and $1 + 5 + 5^2 = 31$ are not triangular. This concludes the proof.