

# MODS 2019/2

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TWITCH SOLVES ISL

Episode 25

## Problem

Find all positive integers  $b$  such that

$$1 + b + b^2 + \cdots + b^e$$

is a triangular number for every positive integer  $e$ .

## Video

<https://youtu.be/EE3z66dcd4s>

## Solution

The answer is  $b = 9$  only, which works since

$$1 + 9 + \dots + 9^e = \frac{1}{2} \frac{3^{e+1} + 1}{2} \frac{3^{e+1} - 1}{2}.$$

To prove  $b = 9$  is the only one possible, note that for  $e = 2$  and  $e = 4$  we find that

$$8(b+1)+1 \quad \text{and} \quad 8(b^3+b^2+b+1)+1$$

are perfect squares. Hence the product  $A$  defined by

$$\begin{aligned} A &:= 4096 [8(b+1)+1] [8(b^3+b^2+b+1)+1] \\ &= 4096(8b+9)(8b^3+8b^2+8b+9) \\ &= 4096(64b^4+136b^3+136b^2+144b+81) \\ &= 4096 \cdot \left[ \left( 8b^2 + \frac{17}{2}b + \frac{255}{64} \right)^2 + \frac{4881b}{64} + \frac{266751}{4096} \right] \\ &= (512b^2 + 544b + 255)^2 + (312384b + 266751) \end{aligned}$$

is a perfect square divisible by 4096.

However,  $312384b + 266751 < 33(1024b^2 + 1088b + 533)$  holds for all  $b \geq 10$ . So when  $b \geq 10$  we have  $(512b^2 + 544b + 255)^2 < A < (512b^2 + 544b + 288)^2$ . Since  $A$  was supposed to be a perfect square divisible by 4096, one would have to have  $A = (512b^2 + 544b + 256)^2$ , so

$$\begin{aligned} (512b^2 + 544b + 256)^2 &= (512b^2 + 544b + 255)^2 + (312384b + 266751) \\ \iff 312384b + 266751 &= 1024b^2 + 1088b + 511 \\ \iff 0 &= 1024b^2 - 311296b + 266240 \\ \iff 0 &= b^2 - 304b + 260 \end{aligned}$$

But the discriminant  $304^2 - 4 \cdot 260$  isn't a square, so this is a contradiction.

Hence, it suffices to exclude  $b = 1, 2, \dots, 8$  exhaustively. For  $1+b$  to be a triangular number for  $b \leq 8$ , one needs  $b \in \{2, 5\}$ , but  $1 + 2 + 2^2 = 7$  and  $1 + 5 + 5^2 = 31$  are not triangular. This concludes the proof.