Iran TST 2019/15

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Twitch Solves ISL

Episode 25

Problem

In triangle ABC, M, N and P are midpoints of sides BC, CA and AB. Point K lies on segment NP so that AK bisects $\angle BKC$. Lines MN and BK intersect at E; lines MP and CK intersects at F. Suppose that H be the foot of perpendicular line from A to BC and L the second intersection of circumcircle of triangles AKH and HEF. Prove that MK, EF and HL are concurrent.

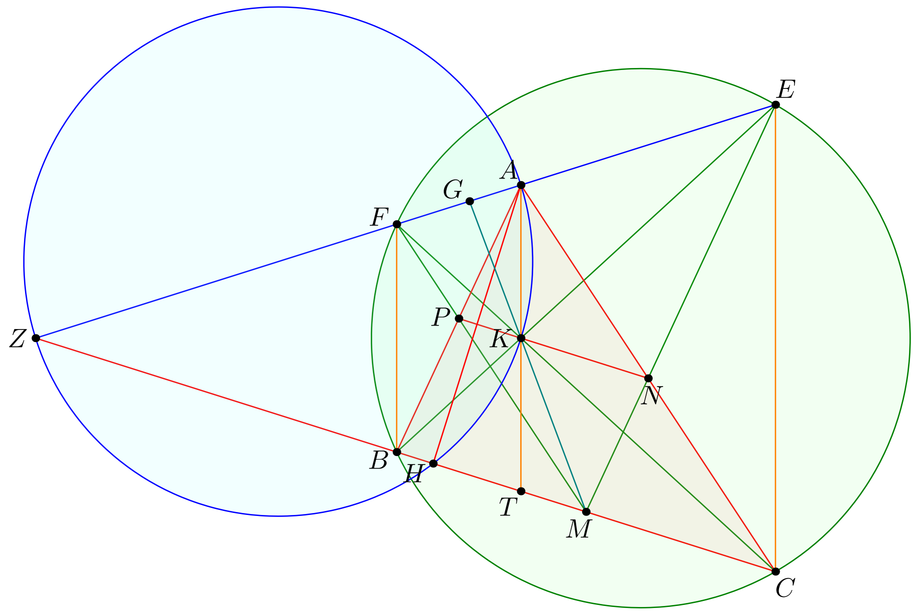
Video

https://youtu.be/3p2rWg5pGwk

Solution

Claim. Points A, E, F are collinear.

Proof. Pappus theorem on BMC and AEF. □



Let $T = \overline{AK} \cap \overline{BC}$.

Claim. Lines BF, CE, AK are pairwise parallel.

Proof. We get BF and CE are parallel by Desargue theorem on $\triangle BPF$ and $\triangle ENC$. However, Desargue theorem on $\triangle BAC$ and $\triangle FTE$ gives BF, AT, CE concurrent. □

Claim. The quadrilateral $BFEC$ is an isosceles trapezoid.

Proof. Since $BK/KC = BT/TC = FA/AE = FK/KE$, we get $BFEC$ is cyclic. □

We let $Z = \overline{EF} \cap \overline{BC}$.

Claim. $ZAKH$ is cyclic, with diameter $\overline{AZ}$.

Proof. Because $\angle ZHA = 90^\circ = \angle ZKA$. □

Finally, let $G = \overline{KM} \cap \overline{EF}$.

Claim. The point G satisfies

$$GE \cdot GF = GA \cdot GZ.$$

Proof. Since $\overline{KM}$ is K-median of $\triangle KBC$, it follows $\overline{KG}$ is K-symmedian of $\triangle KEF$. However, the circle with diameter $\overline{AZ}$ is the K-Apollonian circle of $\triangle KEF$, so it passes through the point Q (not shown) such that $EQFK$ is a harmonic quadrilateral. Since G lies on KQ, the conclusion follows □

Hence the point G is the concurrency point described in the problem, as HL is the radical axis of (AKH) and (HEF).