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Evan Chen

Twitch Solves ISL

Episode 24

Problem

In a $m \times n$ chessboard $(m, n \geq 2)$, some dominoes are placed (without overlap) with each domino covering exactly two adjacent cells. Show that if no more dominoes can be added to the grid, then at least $2/3$ of the chessboard is covered by dominoes.

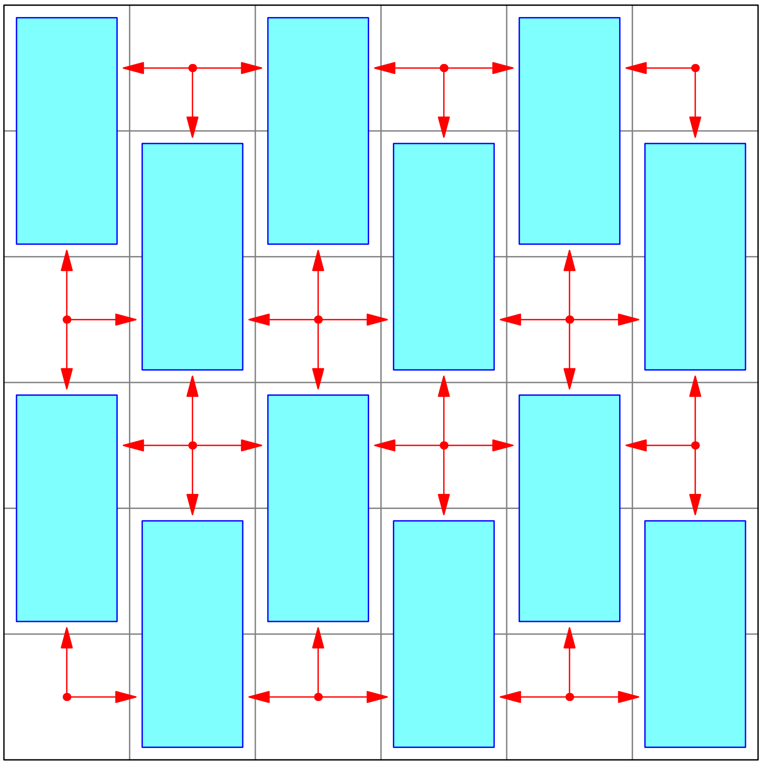
Video

https://youtu.be/wNzJJ2oe_b4

Solution

Assume that some dominoes are placed so that no more dominoes can be added to the grid. We assume $\min(m, n) \geq 3$ for convenience; the case $m = 1$ is straightforward, and the case where $m = 2$ can be dealt with by hand.

From every unoccupied cell not on the border of the board, we emit four lasers from the center of the cell one in each of the four directions; these must each hit the edge of a domino immediately. Cells on the border emit two or three lasers only, according to whether or not they are on the corner.



Claim. Every domino in the grid is hit by at most four lasers; this bound improves to three if the domino is on the border of the grid.

Proof. Each edge of the domino is hit by at most one laser. □

Claim (Laser double counting)**.** The number of dominoes is at least $\frac{mn-1}{3}$.

Moreover, for equality to occur, all four corners must be vacant, and the short end of every domino must either touch the edge of the grid or an adjacent square.

Proof. We make the additional observation that the number of dominoes on the border is at least the number of vacant squares on the border. Indeed, if one travels around the perimeter of the border, between any two vacant squares there is some domino present.

Let D be the number of dominoes, so $mn - 2D$ is the number of vacant squares. Let Y be the number of dominoes which touch the border, and X the number of vacant cells on the border, so that $X \leq Y$. Finally, let $0 \leq e \leq 4$ be the number of vacant corners.

By double counting the lasers, we now have

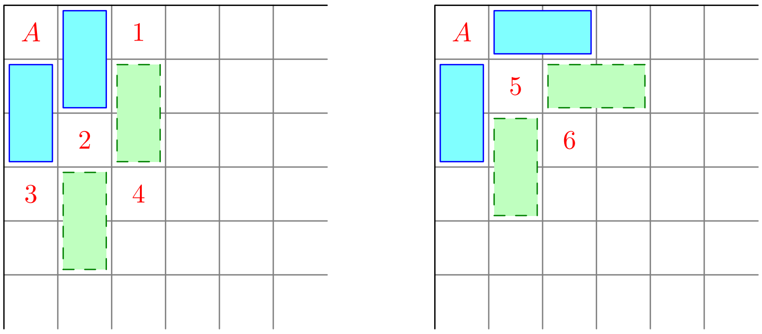
$$4(mn - 2D) - X - e \leq 4D - Y.$$

Since $X \leq Y$ and $e \leq 4$, we recover $3D \geq mn - 1$ as claimed, with equality if $X = Y$, $e = 4$, and every edge of every domino is either on the border or touched by a laser. □

Claim. We can't have $3D = mn - 1$ exactly.

Proof. We take the standing assumptions from the previous claim that all four corners are empty and every edge of every domino is hit by a laser unless it's at the edge of board.

Consider the northwest vacant cell A; it is border by two different dominoes. We consider two different possibilities illustrated below for the layout of the two dominoes touching A.



First case Second case

In first case, the dominoes point the same way, say vertically. Consider the cell marked 1; it must be vacant (since either a horizontal or vertical domino would violate the equality case condition). The cell immediately below 1 must be occupied, and again we see the domino should be vertical. This means the cells marked 2, 3, 4 indicated must be vacant, and we get a vertical domino touching all three. This then lets us repeat the process to find that all dominoes in the first three columns are vertical. We can then extend the process to the right as well; eventually, we find all dominoes are vertical. But then $D > mn/3$ can clearly not hold.

The second case is easier: the cell marked 5 must be vacant, and we must have the two dominoes shown since the short ends of these dominoes cannot touch the blue ones. Then the cells marked 6 is vacant, and we can repeat this argument. Eventually though we reach the edge of the board and a contradiction takes place. $\square$