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Twitch Solves ISL

Episode 24

Problem

Let p be an odd prime and define

$$q(x) = \sum_{k=1}^{p-1} k^{\frac{p-1}{2}} x^k$$

in $\mathbb{F}_p[x]$. Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Video

https://youtu.be/K_YcIS8PW3g

External Link

https://aops.com/community/p13616866

Solution

The answer is $n = \frac{1}{2}(p-1)$.

We use derivatives in the following way.

Claim. Define $q_0 = q$, and $q_{i+1} = x \cdot q_i'$. Suppose n is such that $q_1, \dots, q_{n-1}$ has $x = 1$ as a root, but q_n does not have $x = 1$ as a root. Then n is the multiplicity of $x = 1$ in q.

Proof. This follows from the fact that q_{i+1} will have multiplicity of $x = 1$ one less than in q_i. □

On the other hand, we may explicitly compute

$$q_n(1) = \sum_{k=1}^{p-1} k^{n+\frac{p-1}{2}}.$$

It is a classical fact that the sum of powers vanishes if and only if $p - 1 \nmid n + \frac{p-1}{2}$ (this can be proven by taking a primitive root, say). The smallest n for which this fails is $n = \frac{p-1}{2}$.