

IMO 1988/3

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TWITCH SOLVES ISL

Episode 24

Problem

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned}f(1) &= 1, & f(3) &= 3 \\f(2n) &= f(n) \\f(4n + 1) &= 2f(2n + 1) - f(n) \\f(4n + 3) &= 3f(2n + 1) - 2f(n)\end{aligned}$$

for all positive integers n . Determine with proof the number of positive integers $n \leq 1988$ for which $f(n) = n$.

Video

<https://youtu.be/S6igQE958jQ>

Solution

The main claim is following.

Claim. $f(n)$ is equal to the result when n is written in binary and its digits are reversed.

Proof. Follows directly by induction. □

So the question asks for the number of binary palindromes which are at most $1988 = 11111000100_2$.

For $k = 1, 2, \dots, 10$ there are $2^{\lceil k/2 \rceil - 1}$ binary palindromes with k bits (note the first bit must be 1). For $k = 11$, the number of binary palindromes which are also less than 1988 is $2^5 - 2$ (only 11111011111 and 11111111111 are missing).

Hence the final count is

$$2^0 + 2^0 + 2^1 + 2^1 + 2^2 + 2^2 + 2^3 + 2^3 + 2^4 + 2^4 + (2^5 - 2) = 92.$$