IMO 1988/3

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Twitch Solves ISL

Episode 24

Problem

A function $f \colon \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(1) &= 1, \quad f(3) = 3 \\ f(2n) &= f(n) \\ f(4n+1) &= 2f(2n+1) - f(n) \\ f(4n+3) &= 3f(2n+1) - 2f(n) \end{aligned}$$

for all positive integers n. Determine with proof the number of positive integers $n \le 1988$ for which $f(n) = n$.

Video

https://youtu.be/S6igQE958jQ

External Link

https://aops.com/community/p365112

Solution

The main claim is following.

Claim. $f(n)$ is equal to the result when n is written in binary and its digits are reversed.

Proof. Follows directly by induction. $\square$

So the question asks for the number of binary palindromes which are at most $1988 = 11111000100_2$.

For $k = 1, 2, \dots, 10$ there are $2^{\lceil k/2 \rceil - 1}$ binary palindromes with k bits (note the first bit must be 1). For $k = 11$, the number of binary palindromes which are also less than 1988 is $2^5 - 2$ (only 11111011111 and 11111111111 are missing).

Hence the final count is

$$2^0 + 2^0 + 2^1 + 2^1 + 2^2 + 2^2 + 2^3 + 2^3 + 2^4 + 2^4 + (2^5 - 2) = 92.$$