Shortlist 2009 C4

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Twitch Solves ISL

Episode 23

Problem

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of chessboard, in which each of the 2^m cells along one diagonal forms a separate square of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Video

https://youtu.be/yvW6wnctR-c

External Link

https://aops.com/community/p1932929

Solution

The answer is that $2(2^{m+1} \cdot m) + 4 \cdot 2^m$.

The problem is equivalent to showing that a staircase with $2^m - 1$ cells on the bottom row requires at least $2^{m+1} \cdot m$ cells to cover.

We first prove the lower bound. The general claim is the following:

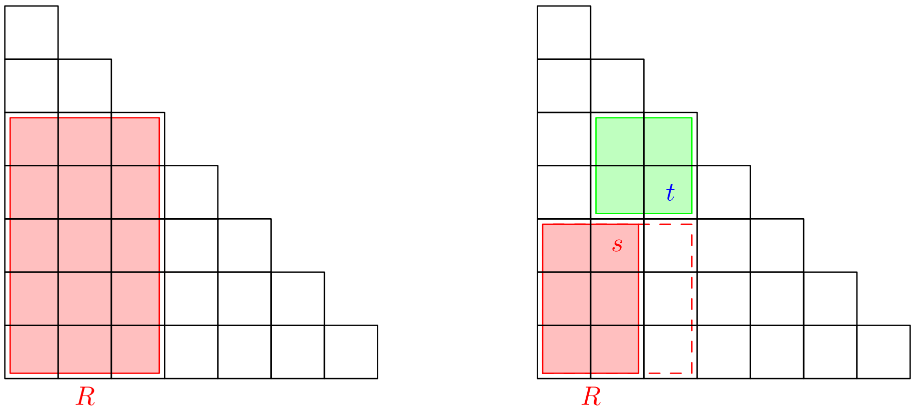
Claim. A staircase with $n - 1$ cells on the bottom row requires at least $f(n) = 2n \log_2 n$ total perimeter to tile with rectangles.

Proof. The proof is by induction on n. Orient the staircase so that the stairs move from the northwest corner to southwest corner. We consider the rectangle R which covers the southwestern cell.

Note that if R touches one of the highest cells in any column, then we can proceed by induction in the following way: we have divided the staircase into two smaller staircases with $a - 1$ and $b - 1$ width, for some a and b, with $a + b = n$. So the total perimeter used must then be at least

$$f(a) + f(b) + 2(a + b) = f(a) + f(b) + 2n.$$

By Jensen inequality on the convex function f, (since $f''(x)$ is a multiple of $1/x$) it follows that we get a lower bound of $2f(n/2) + 2n = f(2n)$ as desired.



If R does *not* have this property, then we will adjust the tiling until it does. Consider the upper right corner s of R. The cell $t = s + (1, 1)$ as shown is covered by some rectangle and we may assume WLOG that the rectangle covering t does not extend south of t, say (as opposed to west; it could be that t does not extend either west or south, or actually the cell t could also be outside the staircase entirely, but the proof remains the same). In that case, we can extend the rectangle R by one cell to the right, as shown, with any rectangles that were once covering those cells being diminished by width 1 to make room. This is possible because no old rectangles could have crossed the northern border of the new R; moreover, the total perimeter cannot increase with such a change. By repeating this adjustment until it is no longer possible, we arrive at the situation described earlier. □

The proof of the lower bound also gives away the construction. When $n = 2^m - 1$, we can split the staircase into an $2^{m-1} \cdot 2^{m-1}$ square plus two staircases with width $2^{m-1} - 1$. So the total perimeter within one staircase is at least $2^{m+1} \cdot m$. The construction matches the lower bound, ending the proof.