SIME 2020/14

Evan Chen

Evan Chen

Twitch Solves ISL

Episode 23

Problem

Let $P(x) = x^3 - 3x^2 + 3$. For how many positive integers $n < 1000$ does there not exist a pair (a, b) of positive integers such that the equation

$$\underbrace{P(P(\dots P}_{a \text{ times}}(x)\dots)) = \underbrace{P(P(\dots P}_{b \text{ times}}(x)\dots))$$

has exactly n distinct real solutions?

Video

https://youtu.be/6RJbk6tTUi0

External Link

https://aops.com/community/p16682015

Solution

The idea is to replace x with the variable $y = 2x + 1$.

To formally, express this consider the substitution $T(y) = 2y + 1$, which is a bijective function. Then

$$P(T(y)) = T\left(4y^3 - 3y\right)$$

Thus, if we define $Q(y) = 4y^3 - 3y$, the problem is actually equivalent to saying that

$$Q(Q(\dots Q(y))) = Q(Q(\dots Q(y)))$$

where Q appears a times on the left and b times on the right.

We now substitute $y = \cos\theta$ for $\theta \in \mathbb{C}$, recognizing Q as the triple-angle formula: $Q(\cos\theta) = \cos 3\theta$. Thus the condition then becomes equivalent to

$$\cos(3^a\theta) = \cos(3^b\theta).$$

Claim. Assume $a < b$. There are $n = 3^b - 3^a + 1$ solutions to the above equation in terms of y, as a function of a and b.

Proof. It is simplest to re-parametrize by $z \in \mathbb{C}$ on the unit circle; the relevant condition is

$$z^{3^a} + z^{-3^a} = z^{3^b} + z^{-3^b}.$$

which simplifies to

$$\left(z^{3^b+3^a} - 1\right)\left(z^{3^b-3^a} - 1\right) = 0.$$

Note there are $\gcd(3^b+3^a, 3^b-3^a) = 2\cdot 3^a$ repeated roots. So in total there are $2\cdot(3^b - 3^a)$ To avoid double counting we only take those z with $\Im z \ge 0$. There are two roots $z = \pm 1$ on the real axis; so this gives the final count $n = 3^b - 3^a + 1$. $\square$

A calculations gives an answer of $999 - (1 + 2 + 3 + 4 + 5) = 984$ now.