

SIME 2020/15

Evan Chen

TWITCH SOLVES ISL

Episode 22

Problem

Triangle ABC has side lengths $AB = 13$, $BC = 14$, and $AC = 15$. Suppose M and N are the midpoints of \overline{AB} and \overline{AC} , respectively. Let P be a point on \overline{MN} , such that if the circumcircles of triangles $\triangle BMP$ and $\triangle CNP$ intersect at a second point Q distinct from P , then PQ is parallel to AB . Calculate AP^2 .

Video

<https://youtu.be/EIz39iVoGcg>

Solution

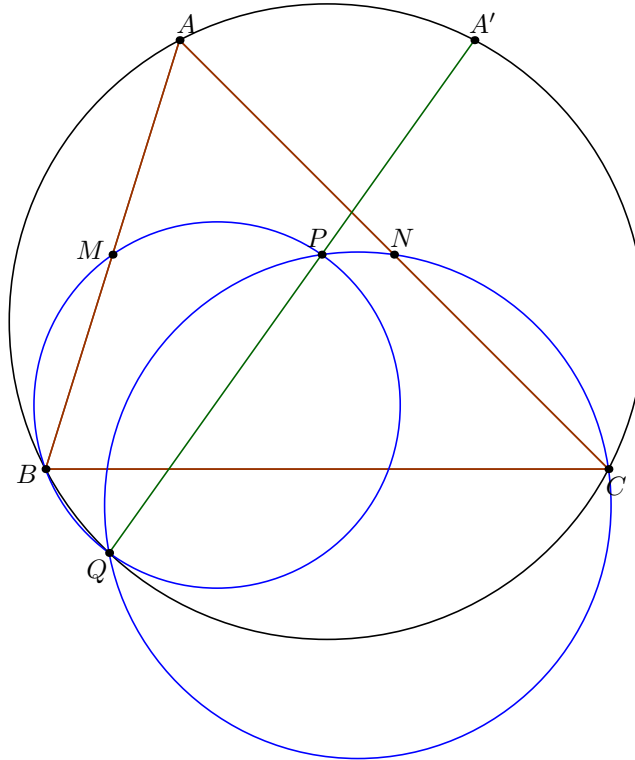
We ignore the condition that $\overline{PQ} \parallel \overline{AB}$ momentarily, and prove the following two claims.

Claim. The point Q lies on (ABC) .

Proof. Miquel point of P, C, B with respect to $\triangle AMN$. □

Claim. The line PQ passes through the reflection of A over the perpendicular bisector of line BC .

Proof. Because $\angle BQP = \angle BMP = \angle NMA = \angle CBA = \angle BCA' = \angle BQA'$. □



We now bring in the condition of the problem. We have $AMPA'$ is a parallelogram in the given problem, so $MP = AA'$, essentially locating P . To compute AA' , we use Ptolemy's theorem by

$$AA' \cdot 14 + 13^2 = 15^2 \implies MP = AA' = 4$$

and a calculation on $\triangle AMN$ now gives $AP^2 = 153/4$.