SIME 2020/15

Evan Chen

Twitch Solves ISL

Episode 22

Problem

Triangle ABC has side lengths $AB = 13$, $BC = 14$, and $AC = 15$. Suppose M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let P be a point on $\overline{MN}$, such that if the circumcircles of triangles $\triangle BMP$ and $\triangle CNP$ intersect at a second point Q distinct from P, then PQ is parallel to AB. Calculate AP^2.

Video

https://youtu.be/EIz39iVoGcg

Solution

We ignore the condition that $\overline{PQ} \parallel \overline{AB}$ momentarily, and prove the following two claims.

Claim. The point Q lies on (ABC).

Proof. Miquel point of P, C, B with respect to $\triangle AMN$. □

Claim. The line PQ passes through the reflection of A over the perpendicular bisector of line BC.

Proof. Because $\measuredangle BQP = \measuredangle BMP = \measuredangle NMA = \measuredangle CBA = \measuredangle BCA' = \measuredangle BQA'$. □

We now bring in the condition of the problem. We have $AMPA'$ is a parallelogram in the given problem, so $MP = AA'$, essentially locating P. To compute AA', we use Ptolemy's theorem by

$$AA' \cdot 14 + 13^2 = 15^2 \implies MP = AA' = 4$$

and a calculation on $\triangle AMN$ now gives $AP^2 = 153/4$.