Croatia TST 2016/2/2

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Twitch Solves ISL

Episode 22

Problem

Let $n > 1$ be an integer. We have an $n \times n$ board with two diagonally opposite corner cells colored in black; the rest is colored white. A legal move consists of picking row or column and inverting the colour of cells which are in that row/column. What is the least additional number of cells that we need to colour black so that we can eventually turn all cells black?

Video

https://youtu.be/BulxRrkmQUQ

External Link

https://aops.com/community/p6260550

Solution

The answer is $2n-4$ additional cells.

We'll do the usual reduction: all the moves commute with each other and doing the same move twice does nothing. Also, it'll be more natural to get all-white to the "starting" state; we'll do so as they are equivalent anyways.

Thus, we may as well assume (by permuting rows/columns)

- we operated on the first a rows;
- we operated on the first b columns.

The given hypothesis is equivalent to saying that if we do this operation on an initially empty board, then we got k black cells, and at least two of them are not in the same row or column. The problem asks for the smallest possible value of $k-2$.

However, the point is that we have the explicit value

$$k = a(n-b) + b(n-a).$$

It will be more economical actually to write

$$n^2 - k = ab + (n-a)(n-b) \le \sqrt{\left(a^2+(n-a)^2\right)\left(b^2+(n-b)^2\right)}.$$

We now analyze two cases:

- If any of a or b are 0 or n, we may directly check we need $k \ge 2n$ in order to have two black cells not in the same row or column.
- Otherwise, $x^2 + (n-x)^2 \le 1 + (n-1)^2$ for $1 \le x \le n-1$, so

  $$n^2 - k \le 1^2 + (n-1)^2$$

  which means $k \ge 2n-2$.

In conclusion, $k - 2 \ge 2n - 4$; and moreover equality occurs at $a = b = 1$, which is checked to indeed work.