

Mexico 2019/6

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TWITCH SOLVES ISL

Episode 21

Problem

Let ABC be a triangle such that $\angle BAC = 45^\circ$. Let H and O be the orthocenter and circumcenter of ABC , respectively. Let ω be the circumcircle of ABC and P the point on ω such that the circumcircle of PBH is tangent to BC . Let X and Y be the circumcenters of PHB and PHC respectively. Let O_1, O_2 be the circumcenters of PXO and PYO respectively. Prove that O_1 and O_2 lie on AB and AC , respectively.

Video

<https://youtu.be/lxuqQaN59tU>

Solution

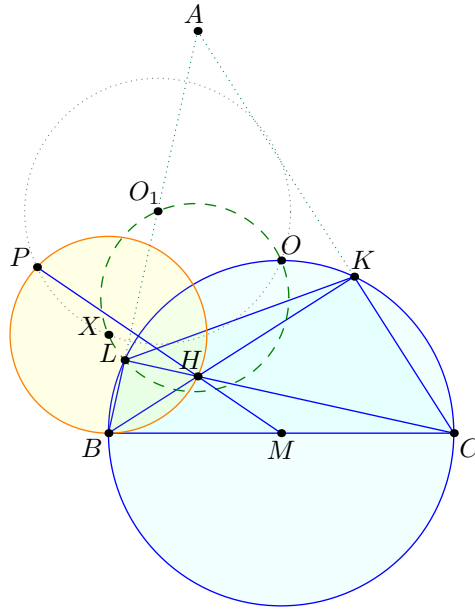
We let \overline{BK} and \overline{CL} be the altitudes of $\triangle ABC$. The circle with diameter \overline{BC} will be denoted by γ ; and we'll denote the center by M .

Claim. The circle (PHC) is also tangent to line BC .

Proof. We are given $\angle PBC = \angle PHB$. Since $\angle BHC = -\angle BAC = -\angle BPC$, it follows $\angle PCB = \angle PHC$ too. \square

Claim. Points P, H, M are collinear. Actually, P is the inverse of H with respect to γ .

Proof. Line PH bisects \overline{BC} by radical axis on (BPH) and (CPH) . Also, $MH \cdot MP = MB^2 = MC^2$. \square



We now actually use the condition that $\angle BAC = 45^\circ$, which is equivalent to $\angle BHC = 135^\circ$ and $\angle BOC = 90^\circ$. This means that O is the arc midpoint of (BC) .

Claim. $LOKH$ is a parallelogram.

Proof. Since arcs KL and OB of γ measure 90° , the arcs BL and OK are equal, hence $\overline{LO} \parallel \overline{BK}$. Similarly $\overline{KO} \parallel \overline{CL}$. \square

Claim. We have $\angle XPO = 45^\circ$.

Proof. Since $\triangle HLK \sim \triangle HBC$, and \overline{HO} bisects \overline{LK} by previous claim, it follows

$$\angle BHM = \angle OHL.$$

Now

$$\begin{aligned} \angle XPO &= \angle XPH + \angle MPO = (90^\circ - \angle HBP) + \angle HOM \\ &= 90^\circ - \angle MBP + \angle MBH + \angle HOM \\ &= 90^\circ - \angle BHM + \angle MBH + \angle HOM \\ &= 90^\circ - \angle OHL + \angle MBH + \angle HOM \\ &= 90^\circ + \angle(LH, HO) + \angle(MB, BH) + \angle(HO, MO) \\ &= 90^\circ + \angle(LH, BH) + \angle(MB, MO) = 45^\circ. \end{aligned} \quad \square$$

Claim. We have X, O_1, O, L, H are concyclic, in the circle with diameter \overline{XO} .

Proof. First, since $\triangle LBH$ is a 45° - 45° - 90° triangle and $XB = XH$, it follows that \overline{XL} is the perpendicular bisector of \overline{BH} . Hence, $\angle XLO = 90^\circ$.

On the other hand, since $\angle XPO = 45^\circ$, we have $\angle XO_1O = 90^\circ$. This implies concyclic. \square

Finally, $\angle BLO = 135^\circ$ and

$$\angle OLO_1 = \angle OXO_1 = 90^\circ - \angle XPO = 45^\circ$$

this implies O_1 lies on line BL , ergo lies on line AB .