Mexico 2019/6

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Twitch Solves ISL

Episode 21

Problem

Let ABC be a triangle such that $\angle BAC = 45^\circ$. Let H and O be the orthocenter and circumcenter of ABC, respectively. Let ω be the circumcircle of ABC and P the point on ω such that the circumcircle of PBH is tangent to BC. Let X and Y be the circumcenters of PHB and PHC respectively. Let O_1, O_2 be the circumcenters of PXO and PYO respectively. Prove that O_1 and O_2 lie on AB and AC, respectively.

Video

https://youtu.be/lxuqQaN59tU

External Link

https://aops.com/community/p13469495

Solution

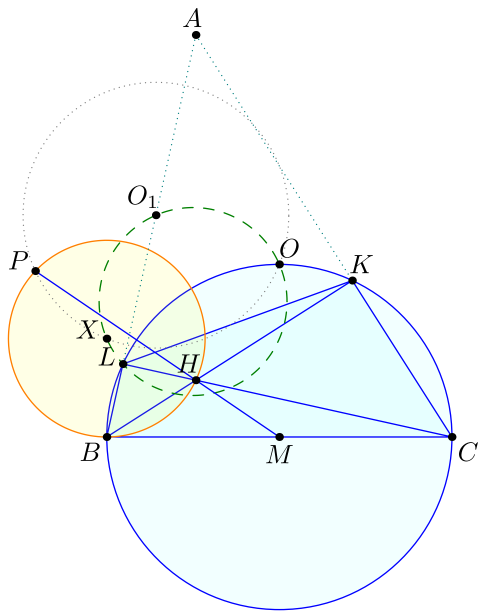
We let $\overline{BK}$ and $\overline{CL}$ be the altitudes of $\triangle ABC$. The circle with diameter $\overline{BC}$ will be denoted by γ; and we'll denote the center by M.

Claim. The circle (PHC) is also tangent to line BC.

Proof. We are given $\measuredangle PBC = \measuredangle PHB$. Since $\measuredangle BHC = -\measuredangle BAC = -\measuredangle BPC$, it follows $\measuredangle PCB = \measuredangle PHC$ too. □

Claim. Points P, H, M are collinear. Actually, P is the inverse of H with respect to γ.

Proof. Line PH bisects $\overline{BC}$ by radical axis on (BPH) and (CPH). Also, $MH \cdot MP = MB^2 = MC^2$. □



We now actually use the condition that $\measuredangle BAC = 45^\circ$, which is equivalent to $\measuredangle BHC = 135^\circ$ and $\measuredangle BOC = 90^\circ$. This means that O is the arc midpoint of (BC).

Claim. $LOKH$ is a parallelogram.

Proof. Since arcs KL and OB of γ measure 90°, the arcs BL and OK are equal, hence $\overline{LO} \parallel \overline{BK}$. Similarly $\overline{KO} \parallel \overline{CL}$. □

Claim. We have $\measuredangle XPO = 45^\circ$.

Proof. Since $\triangle HLK \sim \triangle HBC$, and $\overline{HO}$ bisects $\overline{LK}$ by previous claim, it follows

$$\measuredangle BHM = \measuredangle OHL.$$

Now

$$\begin{aligned}
\measuredangle XPO = \measuredangle XPH + \measuredangle MPO &= (90^\circ - \measuredangle HBP) + \measuredangle HOM \\
&= 90^\circ - \measuredangle MBP + \measuredangle MBH + \measuredangle HOM \\
&= 90^\circ - \measuredangle BHM + \measuredangle MBH + \measuredangle HOM \\
&= 90^\circ - \measuredangle OHL + \measuredangle MBH + \measuredangle HOM \\
&= 90^\circ + \measuredangle(LH, HO) + \measuredangle(MB, BH) + \measuredangle(HO, MO) \\
&= 90^\circ + \measuredangle(LH, BH) + \measuredangle(MB, MO) = 45^\circ.
\end{aligned}$$ □

Claim. We have X, O_1, O, L, H are concyclic, in the circle with diameter $\overline{XO}$.

Proof. First, since $\triangle LBH$ is a $45°$-$45°$-$90°$ triangle and $XB = XH$, it follows that $\overline{XL}$ is the perpendicular bisector of $\overline{BH}$. Hence, $\measuredangle XLO = 90°$.

On the other hand, since $\measuredangle XPO = 45°$, we have $\measuredangle XO_1O = 90°$. This implies concyclic. □

Finally, $\measuredangle BLO = 135°$ and

$$\measuredangle OLO_1 = \measuredangle OXO_1 = 90° - \measuredangle XPO = 45°$$

this implies O_1 lies on line BL, ergo lies on line AB.