IMO 1979/3

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Twitch Solves ISL

Episode 21

Problem

Two circles in a plane intersect and A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each travelling along its own circle in the same direction. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

Video

https://youtu.be/cavQ_A0QS8o

External Link

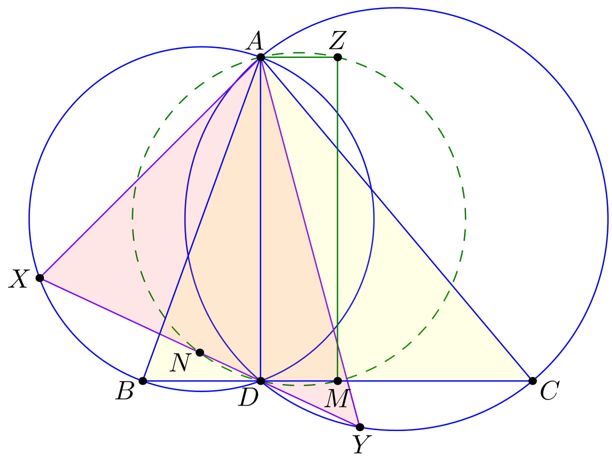
https://aops.com/community/p367352

Solution

Let B and C be the antipodes of A on the two circles and let D be the foot of the altitude from A, which is the other intersection point of the two circles. Also, let M be the midpoint of $\overline{BC}$, and construct rectangle $ADMZ$. Our claim is that Z is the fixed point.

We let X and Y be the two points; by the condition the angles are the same. So we have a spiral similarity

$$\triangle AXY \sim \triangle ABC.$$



Now let N be the midpoint of $\overline{XY}$. By spiral similarity, since N maps to M, it follows M, N, A, and D are cyclic too. So actually N lies on the circumcircle of rectangle $ADMZ$, meaning $\overline{ZN} \perp \overline{XY}$, hence $ZX = ZY$ as needed.

Remark. The special point Z can be identified by selecting the special case $X \to A$, $Y \to A$ and $X = B$, $Y = C$.