

IMO 1979/3

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TWITCH SOLVES ISL

Episode 21

Problem

Two circles in a plane intersect and A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each travelling along its own circle in the same direction. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P .

Video

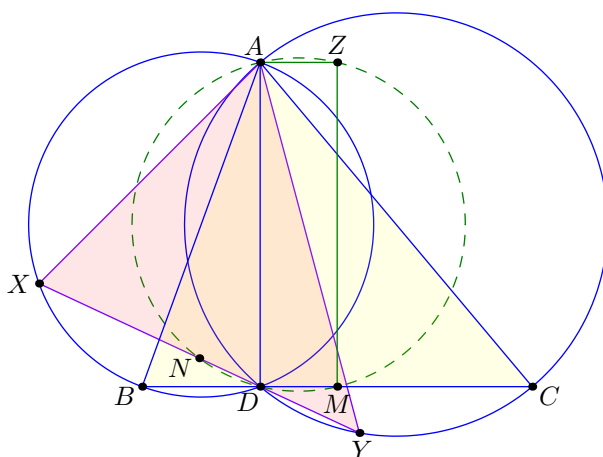
https://youtu.be/cavQ_A0QS8o

Solution

Let B and C be the antipodes of A on the two circles and let D be the foot of the altitude from A , which is the other intersection point of the two circles. Also, let M be the midpoint of \overline{BC} , and construct rectangle $ADMZ$. Our claim is that Z is the fixed point.

We let X and Y be the two points; by the condition the angles are the same. So we have a spiral similarity

$$\triangle AXY \sim \triangle ABC.$$



Now let N be the midpoint of \overline{XY} . By spiral similarity, since N maps to M , it follows $M, N, A,$ and D are cyclic too. So actually N lies on the circumcircle of rectangle $ADMZ$, meaning $\overline{ZN} \perp \overline{XY}$, hence $ZX = ZY$ as needed.

Remark. The special point Z can be identified by selecting the special case $X \rightarrow A,$ $Y \rightarrow A$ and $X = B, Y = C$.