

# IMO 1979/3

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TWITCH SOLVES ISL

Episode 21

## Problem

Two circles in a plane intersect and  $A$  is one of the points of intersection. Starting simultaneously from  $A$  two points move with constant speed, each travelling along its own circle in the same direction. The two points return to  $A$  simultaneously after one revolution. Prove that there is a fixed point  $P$  in the plane such that the two points are always equidistant from  $P$ .

## Video

[https://youtu.be/cavQ\\_A0QS8o](https://youtu.be/cavQ_A0QS8o)

## External Link

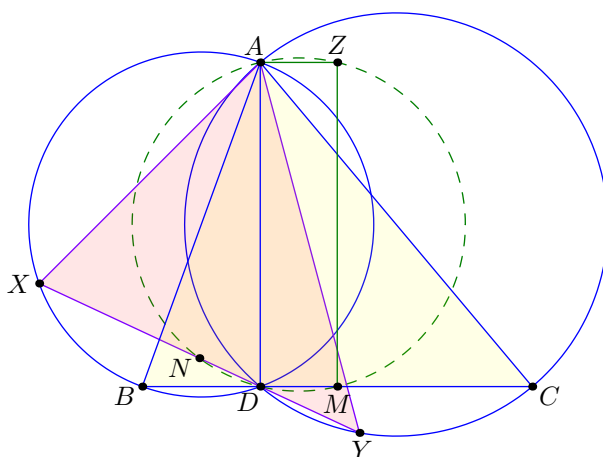
<https://aops.com/community/p367352>

## Solution

Let  $B$  and  $C$  be the antipodes of  $A$  on the two circles and let  $D$  be the foot of the altitude from  $A$ , which is the other intersection point of the two circles. Also, let  $M$  be the midpoint of  $\overline{BC}$ , and construct rectangle  $ADMZ$ . Our claim is that  $Z$  is the fixed point.

We let  $X$  and  $Y$  be the two points; by the condition the angles are the same. So we have a spiral similarity

$$\triangle AXY \sim \triangle ABC.$$



Now let  $N$  be the midpoint of  $\overline{XY}$ . By spiral similarity, since  $N$  maps to  $M$ , it follows  $M, N, A,$  and  $D$  are cyclic too. So actually  $N$  lies on the circumcircle of rectangle  $ADMZ$ , meaning  $\overline{ZN} \perp \overline{XY}$ , hence  $ZX = ZY$  as needed.

**Remark.** The special point  $Z$  can be identified by selecting the special case  $X \rightarrow A,$   $Y \rightarrow A$  and  $X = B, Y = C$ .