Romania JBMO TST 2019/3.1

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Twitch Solves ISL

Episode 19

Problem

Find all pairs (a, b) of relatively prime positive integers such that

$$a^2 + b = (a - b)^3.$$

Video

https://youtu.be/yAU826DfIdo

Solution

The only answer is $(a, b) = (5, 2)$ which works.

To see it is the only one, note that taking modulo a and b respectively gives:

- $b \equiv (-b)^3 \pmod a \implies a \mid b^2 + 1$
- $a^2 \equiv a^3 \pmod b \implies b \mid a - 1$.

Let us set $a = k \cdot b + 1$. Then the first equation gives

$$kb + 1 \mid k(b^2 + 1) = k \cdot kb + k \implies kb + 1 \mid -b + k.$$

For size reasons this forces $k = b$, else $|kb + 1| > |k - b| > 0$.

In other words, we need $a = b^2 + 1$. So

$$(b^2 + 1)^2 + b = (b^2 + 1 - b)^3.$$

Now a calculation shows

$$(b^2 - b + 1)^3 - \left[(b^2 + 1)^2 + b\right] = b(b - 2)(b^2 + 1)(b^2 + b - 2)$$

so $b = 2$ is the only integer solution.