

KJMO 2012/1

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TWITCH SOLVES ISL

Episode 19

Problem

Let a, b, c be positive real numbers satisfying $ab + bc + ca = 1$. Prove that

$$\frac{a+b}{\sqrt{ab(1-ab)}} + \frac{b+c}{\sqrt{bc(1-bc)}} + \frac{c+a}{\sqrt{ca(1-ca)}} \leq \frac{\sqrt{2}}{abc}$$

Video

<https://youtu.be/lC1g-01Me-w>

Solution

Since $1 - ab = c(a + b)$ the inequality rewrites as

$$\sum_{\text{cyc}} \sqrt{a+b} \leq \sqrt{\frac{2}{abc}}(ab + bc + ca)$$

or simply

$$\sum_{\text{cyc}} \sqrt{\frac{a+b}{2}} \leq \sum_{\text{cyc}} \sqrt{\frac{ab}{c}}$$

If we let $x = \sqrt{\frac{bc}{a}}$, etc., then the inequality rewrites as

$$\sum_{\text{cyc}} \sqrt{\frac{x(y+z)}{2}} \leq x + y + z.$$

This follows by using AM-GM

$$\sqrt{\frac{x(y+z)}{2}} \leq \frac{1}{2} \left[x + \frac{y+z}{2} \right]$$

and summing cyclically.