Shortlist 2002 C7

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Episode 18

Problem

Among a group of 120 people, some pairs are friends. A weak quartet is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets ?

Video

https://youtu.be/4AxsoRBAYE0

External Link

https://aops.com/community/p118720

Solution

The answer is 4769280 achieved when the graph G is five disjoint copies of K_{24}.

The following claim reduces the problem to an annoying (but routine) calculation:

Claim. The optimum is achieved when G consists of the disjoint union of several cliques.

Proof. The proof goes by Zykov symmetrization. Assume that G is chosen so the number of weak quartets is maximized.

Throughout the process, we annotate every vertex v of G with the number of weak quartets that G is in, say $n(v)$. Now the main observation is that: if $n(v) \ge n(v')$, then deleting v' and replacing it with a clone of v changes the number of weak quartets in the graph by $n(v) - n(v')$. Indeed,

- Any weak quartet that involved v' but not v is destroyed;
- Any weak quartet that involved v but not v' is cloned;
- Any weak quartet that involved both v and v' is unaffected (as is any weak quartet which involved neither).

Since G was picked maximally, that means in fact $n(v) = n(v')$ for any v and v' in the same connected component of G. Hence we can do the cloning operation between any pair of adjacent vertices v and v', with no change to the number of weak quartets.

Moreover, doing so won't change $n(w)$ for any other vertices w which remain in the same connected component as the cloned v. After all, if it did, then we could perform the operation in a way that strictly increases the number of weak quartets, contradicting again the maximality of G.

So, for each component C of G, we pick any vertex v and clone it everywhere. This could break off C into multiple components, which is okay; we then handle those components by the same procedure later on. Repeating this procedure several times we eventually arrive at G a disjoint union of cliques. □

The rest of the calculation is only outlined below. If G has cliques of size $a_1, \dots, a_n$ then the number of weak quartets is exactly

$$\sum_1^n \binom{a_i}{2} \sum_{\substack{1 \le j < k \le n \\ j,k \ne i}} a_j a_k$$

and a calculation shows we want these to be as close as equal as possible. Up to integer rounding issues, this means we have

$$n \cdot \binom{120/n}{2} \binom{n-1}{2} \cdot \left(\frac{120}{n} \right)^2$$

which turns out to be maximal at $n = 5$ giving the answer earlier.