# Shortlist 2002 C7 <br> Evan Chen 

Twitch Solves ISL
Episode 18

## Problem

Among a group of 120 people, some pairs are friends. A weak quartet is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets ?

## Video

https://youtu.be/4AxsoRBAYE0

## External Link

https://aops.com/community/p118720

## Solution

The answer is 4769280 achieved when the graph $G$ is five disjoint copies of $K_{24}$.
The following claim reduces the problem to an annoying (but routine) calculation:
Claim. The optimum is achieved when $G$ consists of the disjoint union of several cliques.
Proof. The proof goes by Zykov symmetrization. Assume that $G$ is chosen so the number of weak quartets is maximized.

Throughout the process, we annotate every vertex $v$ of $G$ with the number of weak quartets that $G$ is in, say $n(v)$. Now the main observation is that: if $n(v) \geq n\left(v^{\prime}\right)$, then deleting $v^{\prime}$ and replacing it with a clone of $v$ changes the number of weak quartets in the graph by $n(v)-n\left(v^{\prime}\right)$. Indeed,

- Any weak quartet that involved $v^{\prime}$ but not $v$ is destroyed;
- Any weak quartet that involved $v$ but not $v^{\prime}$ is cloned;
- Any weak quartet that involved both $v$ and $v^{\prime}$ is unaffected (as is any weak quartet which involved neither).

Since $G$ was picked maximally, that means in fact $n(v)=n\left(v^{\prime}\right)$ for any $v$ and $v^{\prime}$ in the same connected component of $G$. Hence we can do the cloning operation between any pair of adjacent vertices $v$ and $v^{\prime}$, with no change to the number of weak quartets.

Moreover, doing so won't change $n(w)$ for any other vertices $w$ which remain in the same connected component as the cloned $v$. After all, if it did, then we could perform the operation in a way that strictly increases the number of weak quartets, contradicting again the maximality of $G$.

So, for each component $C$ of $G$, we pick any vertex $v$ and clone it everywhere. This could break off $C$ into multiple components, which is okay; we then handle those components by the same procedure later on. Repeating this procedure several times we eventually arrive at $G$ a disjoint union of cliques.

The rest of the calculation is only outlined below. If $G$ has cliques of size $a_{1}, \ldots, a_{n}$ then the number of weak quartets is exactly

$$
\sum_{1}^{n}\binom{a_{i}}{2} \sum_{\substack{1 \leq j<k \leq n \\ j, k \neq i}} a_{j} a_{k}
$$

and a calculation shows we want these to be as close as equal as possible. Up to integer rounding issues, this means we have

$$
n \cdot\binom{120 / n}{2}\binom{n-1}{2} \cdot\left(\frac{120}{n}\right)^{2}
$$

which turns out to be maximal at $n=5$ giving the answer earlier.

