

Shortlist 2004 G6

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TWITCH SOLVES ISL

Episode 17

Problem

Let \mathcal{P} be a convex polygon. Prove that there exists a convex hexagon that is contained in \mathcal{P} and whose area is at least $\frac{3}{4}$ of the area of the polygon \mathcal{P} .

Video

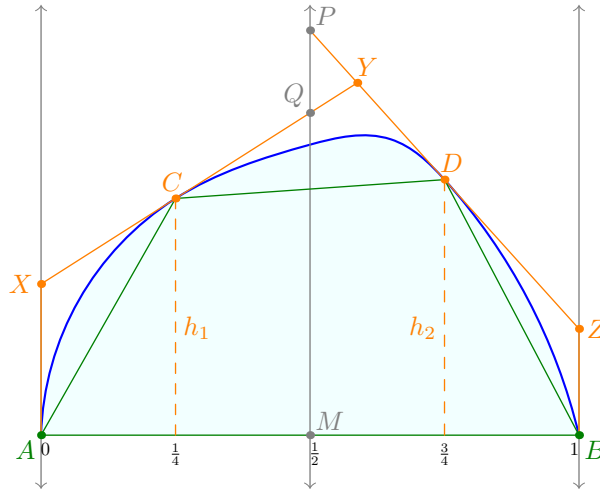
<https://youtu.be/okLAo36yxdk>

Solution

We are going to solve the problem when \mathcal{P} is replaced by a differentiable convex curve. (The proof requires only trivial modifications for a polygon; in any case, polygons can approximate convex curves arbitrarily well and vice versa.)

We start by committing to take the longest segment AB as a major diagonal of our hexagon. Therefore, this cuts \mathcal{P} into two halves and we deal with each half separately.

Orient AB on the x -axis with $A = (0,0)$ and $B = (1,0)$ and consider the region above the x -axis. Let C and D be the points on the curve with x -coordinates $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Then the lines $x = 0$ and $x = 1$, together with the tangents at C and D , determine a pentagon $AXYZB$ that encloses \mathcal{P} (since \mathcal{P} is convex), as shown.



The claim is that $ACDB$ fits the bill:

Claim. We have

$$[ACDB] \geq \frac{3}{4}[AXYZB].$$

Proof. Let h_1 and h_2 be the y -coordinates of C and D . Also, as depicted, let $x = 1/2$ meet YDZ at P and XCX at Q , and let $M = (1/2, 0)$. Then

$$\begin{aligned} \frac{4}{3} \cdot [ACDB] &= \frac{4}{3} \left(\frac{1}{8}h_1 + \frac{1}{4}(h_1 + h_2) + h_2 \right) = h_1 + h_2 \\ &= [AXQM] + [BZPM] \geq [AXYZB]. \end{aligned}$$

Note that equality when $P = Q = Y$. □

Repeating the same proof for the bottom half of \mathcal{P} exhibits the desired hexagon.