Shortlist 2004 G6

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Twitch Solves ISL

Episode 17

Problem

Let $\mathcal{P}$ be a convex polygon. Prove that there exists a convex hexagon that is contained in $\mathcal{P}$ and whose area is at least $\frac{3}{4}$ of the area of the polygon $\mathcal{P}$.

Video

https://youtu.be/okLAo36yxdk

External Link

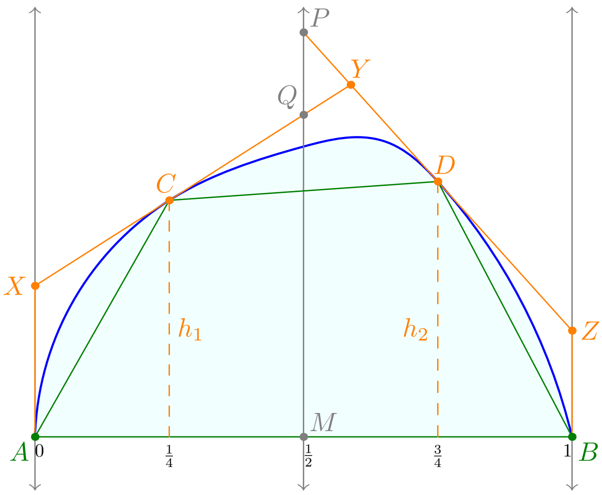
https://aops.com/community/p143293

Solution

We are going to solve the problem when $\mathcal{P}$ is replaced by a differentiable convex curve. (The proof requires only trivial modifications for a polygon; in any case, polygons can approximate convex curves arbitrarily well and vice versa.)

We start by committing to take the longest segment AB as a major diagonal of our hexagon. Therefore, this cuts $\mathcal{P}$ into two halves and we deal with each half separately.

Orient AB on the x-axis with $A = (0,0)$ and $B = (1,0)$ and consider the region above the x-axis. Let C and D be the points on the curve with x-coordinates $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Then the lines $x = 0$ and $x = 1$, together with the tangents at C and D, determine a pentagon $AXYZB$ that encloses $\mathcal{P}$ (since $\mathcal{P}$ is convex), as shown.



The claim is that $ACDB$ fits the bill:

Claim. We have
$$[ACDB] \geq \frac{3}{4}[AXYZB].$$

Proof. Let h_1 and h_2 be the y-coordinates of C and D. Also, as depicted, let $x = 1/2$ meet YDZ at P and XCY at Q, and let $M = (1/2, 0)$. Then

$$\begin{aligned} \frac{4}{3} \cdot [ACDB] &= \frac{4}{3}\left(\frac{1}{8}h_1 + \frac{1}{4}(h_1 + h_2) + \frac{1}{8}h_2\right) = \frac{1}{2}(h_1 + h_2) \\ &= [AXQM] + [BZPM] \geq [AXYZB]. \end{aligned}$$

Note that equality when $P = Q = Y$. $\square$

Repeating the same proof for the bottom half of $\mathcal{P}$ exhibits the desired hexagon.