

# USAMO 2020/6

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TWITCH SOLVES ISL

Episode 16

## Problem

Let  $n \geq 2$  be an integer. Let  $x_1 \geq x_2 \geq \cdots \geq x_n$  and  $y_1 \geq y_2 \geq \cdots \geq y_n$  be  $2n$  real numbers such that

$$\begin{aligned} 0 &= x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n, \\ \text{and } 1 &= x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2. \end{aligned}$$

Prove that

$$\sum_{i=1}^n (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

## Video

<https://youtu.be/r7j0oRtpErA>

## External Link

<https://aops.com/community/p15953303>

## Solution

We present two approaches. In both approaches, it's helpful motivation that for even  $n$ , equality occurs at

$$(x_i) = \left( \underbrace{\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\frac{1}{\sqrt{n}}, \dots, -\frac{1}{\sqrt{n}}}_{n/2} \right)$$

$$(y_i) = \left( \frac{n-1}{\sqrt{n(n-1)}}, \underbrace{-\frac{1}{\sqrt{n(n-1)}}, \dots, -\frac{1}{\sqrt{n(n-1)}}}_{n-1} \right)$$

**First approach (expected value).** For a permutation  $\sigma$  on  $\{1, 2, \dots, n\}$  we define

$$S_\sigma = \sum_{i=1}^n x_i y_{\sigma(i)}.$$

**Claim.** For random permutations  $\sigma$ ,  $\mathbb{E}[S_\sigma] = 0$  and  $\mathbb{E}[S_\sigma^2] = \frac{1}{n-1}$ .

*Proof.* The first one is clear.

Since  $\sum_{i < j} 2x_i x_j = -1$ , it follows that (for fixed  $i$  and  $j$ ),  $\mathbb{E}[y_{\sigma(i)} y_{\sigma(j)}] = -\frac{1}{n(n-1)}$ . Thus

$$\sum_i x_i^2 \cdot \mathbb{E}[y_{\sigma(i)}^2] = \frac{1}{n}$$

$$2 \sum_{i < j} x_i x_j \cdot \mathbb{E}[y_{\sigma(i)} y_{\sigma(j)}] = (-1) \cdot \frac{1}{n(n-1)}$$

the conclusion follows.  $\square$

**Claim** (A random variable in  $[0, 1]$  has variance at most  $1/4$ ). If  $A$  is a random variable with mean  $\mu$  taking values in the closed interval  $[m, M]$  then

$$\mathbb{E}[(A - \mu)^2] \leq \frac{1}{4}(M - m)^2.$$

*Proof.* By shifting and scaling, we may assume  $m = 0$  and  $M = 1$ , so  $A \in [0, 1]$  and hence  $A^2 \leq A$ . Then

$$\mathbb{E}[(A - \mu)^2] = \mathbb{E}[A^2] - \mu^2 \leq \mathbb{E}[A] - \mu^2 = \mu - \mu^2 \leq \frac{1}{4}.$$

This concludes the proof.  $\square$

Thus the previous two claims together give

$$\max_{\sigma} S_{\sigma} - \min_{\sigma} S_{\sigma} \geq \sqrt{\frac{4}{n-1}} = \frac{2}{\sqrt{n-1}}.$$

But  $\sum_i x_i y_i = \max_{\sigma} S_{\sigma}$  and  $\sum_i x_i y_{n+1-i} = \min_{\sigma} S_{\sigma}$  by rearrangement inequality and therefore we are done.

**Outline of second approach (by convexity, due to Alex Zhai).** We will instead prove a converse result: given the hypotheses

- $x_1 \geq \cdots \geq x_n$
- $y_1 \geq \cdots \geq y_n$
- $\sum_i x_i = \sum_i y_i = 0$
- $\sum_i x_i y_i - \sum_i x_i y_{n+1-i} = \frac{2}{\sqrt{n-1}}$

we will prove that  $\sum x_i^2 \sum y_i^2 \leq 1$ .

Fix the choice of  $y$ 's. We see that we are trying to maximize a convex function in  $n$  variables  $(x_1, \dots, x_n)$  over a convex domain (actually the intersection of two planes with several half planes). So a maximum can only happen at the boundaries: when at most two of the  $x$ 's are different.

An analogous argument applies to  $y$ . In this way we find that it suffices to consider situations where  $x_\bullet$  takes on at most two different values. The same argument applies to  $y_\bullet$ .

At this point the problem can be checked directly.