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Twitch Solves ISL

Episode 16

Problem

Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be $2n$ real numbers such that

$$\begin{aligned} 0 &= x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n, \\ \text{and} \quad 1 &= x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2. \end{aligned}$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

Video

https://youtu.be/r7j0oRtpErA

External Link

https://aops.com/community/p15953303

Solution

We present two approaches. In both approaches, it's helpful motivation that for even n, equality occurs at

$$(x_i) = \Big(\underbrace{\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\frac{1}{\sqrt{n}}, \dots, -\frac{1}{\sqrt{n}}}_{n/2} \Big)$$

$$(y_i) = \Big(\frac{n-1}{\sqrt{n(n-1)}}, \underbrace{-\frac{1}{\sqrt{n(n-1)}}, \dots, -\frac{1}{\sqrt{n(n-1)}}}_{n-1} \Big)$$

First approach (expected value). For a permutation σ on $\{1, 2, \dots, n\}$ we define

$$S_\sigma = \sum_{i=1}^{n} x_i y_{\sigma(i)}.$$

Claim. For random permutations σ, $\mathbb{E}[S_\sigma] = 0$ and $\mathbb{E}[S_\sigma^2] = \frac{1}{n-1}$.

Proof. The first one is clear.

Since $\sum_{i<j} 2x_i x_j = -1$, it follows that (for fixed i and j), $\mathbb{E}[y_{\sigma(i)} y_{\sigma(j)}] = -\frac{1}{n(n-1)}$, Thus

$$\sum_i x_i^2 \cdot \mathbb{E}\left[y_{\sigma(i)}^2 \right] = \frac{1}{n}$$

$$2 \sum_{i<j} x_i x_j \cdot \mathbb{E}\left[y_{\sigma(i)} y_{\sigma(j)} \right] = (-1) \cdot \frac{1}{n(n-1)}$$

the conclusion follows. □

Claim (A random variable in $[0,1]$ has variance at most $1/4$)**.** If A is a random variable with mean μ taking values in the closed interval $[m, M]$ then

$$\mathbb{E}[(A-\mu)^2] \le \frac{1}{4}(M-m)^2.$$

Proof. By shifting and scaling, we may assume $m = 0$ and $M = 1$, so $A \in [0,1]$ and hence $A^2 \le A$. Then

$$\mathbb{E}[(A-\mu)^2] = \mathbb{E}[A^2] - \mu^2 \le \mathbb{E}[A] - \mu^2 = \mu - \mu^2 \le \frac{1}{4}.$$

This concludes the proof. □

Thus the previous two claims together give

$$\max_\sigma S_\sigma - \min_\sigma S_\sigma \ge \sqrt{\frac{4}{n-1}} = \frac{2}{\sqrt{n-1}}.$$

But $\sum_i x_i y_i = \max_\sigma S_\sigma$ and $\sum_i x_i y_{n+1-i} = \min_\sigma S_\sigma$ by rearrangement inequality and therefore we are done.

Outline of second approach (by convexity, due to Alex Zhai). We will instead prove a converse result: given the hypotheses

- $x_1 \ge \cdots \ge x_n$
- $y_1 \ge \cdots \ge y_n$
- $\sum_i x_i = \sum_i y_i = 0$
- $\sum_i x_i y_i - \sum_i x_i y_{n+1-i} = \frac{2}{\sqrt{n-1}}$

we will prove that $\sum x_i^2 \sum y_i^2 \le 1$.

Fix the choice of y's. We see that we are trying to maximize a convex function in n variables $(x_1, \dots, x_n)$ over a convex domain (actually the intersection of two planes with several half planes). So a maximum can only happen at the boundaries: when at most two of the x's are different.

An analogous argument applies to y. In this way we find that it suffices to consider situations where $x_\bullet$ takes on at most two different values. The same argument applies to $y_\bullet$.

At this point the problem can be checked directly.