USAMO 2020/4

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Twitch Solves ISL

Episode 16

Problem

Suppose that (a_1, b_1), (a_2, b_2), $\dots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Video

https://youtu.be/r7j0oRtpErA?

Solution

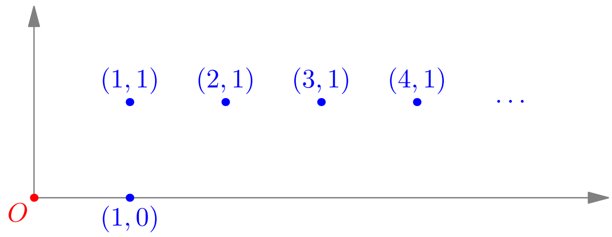
The answer is 197. In general, if 100 is replaced by $n \ge 2$ the answer is $2n-3$.

The idea is that if we let $P_i = (a_i, b_i)$ be a point in the coordinate plane, and let $O = (0,0)$ then we wish to maximize the number of triangles $\triangle OP_iP_j$ which have area $1/2$. Call such a triangle *good*.

Construction of 197 points: It suffices to use the points $(1,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $\dots$, $(99,1)$ as shown. Notice that:

- There are 98 good triangles with vertices $(0,0)$, $(k,1)$ and $(k+1,1)$ for $k = 1, \dots, 98$.
- There are 99 good triangles with vertices $(0,0)$, $(1,0)$ and $(k,1)$ for $k = 1, \dots, 99$.

This is a total of $98 + 99 = 197$ triangles.

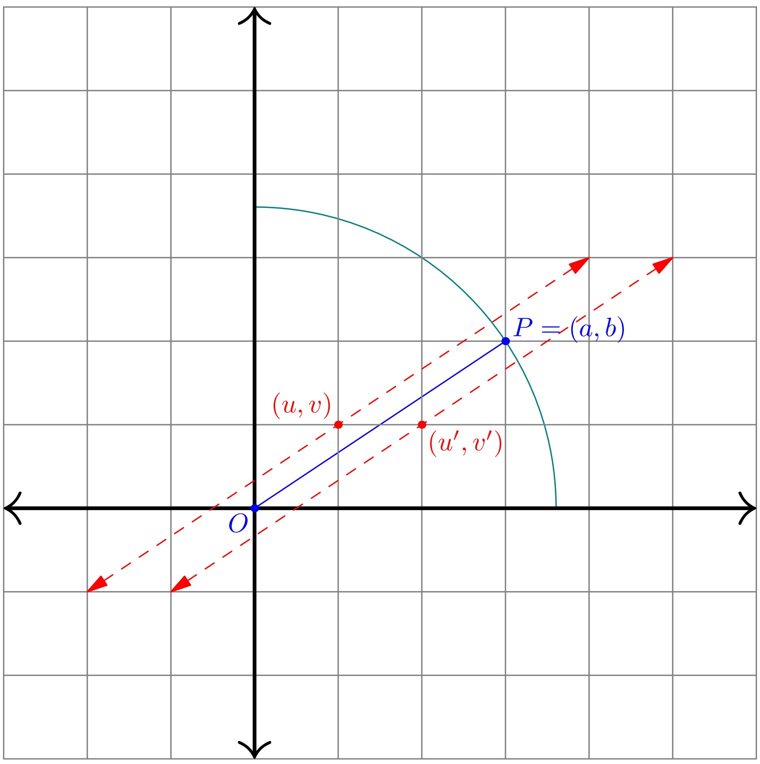


Proof that 197 points is optimal: We proceed by induction on n to show the bound of $2n-3$. The base case $n = 2$ is evident.

For the inductive step, suppose (without loss of generality) that the point $P = P_n = (a,b)$ is the farthest away from the point O among all points.

Claim. This farthest point $P = P_n$ is part of at most two good triangles.

Proof. We must have $\gcd(a,b) = 1$ for P to be in any good triangles at all, since otherwise any divisor of $\gcd(a,b)$ also divides $2[OPQ]$. Now, we consider the locus of all points Q for which $[OPQ] = 1/2$. It consists of two parallel lines passing with slope OP, as shown.



Since $\gcd(a, b) = 1$, see that only two lattice points on this locus actually lie inside the quarter-circle centered at O with radius OP. Indeed if one of the points is (u, v) then the others on the line are $(u \pm a, v \pm b)$ where the signs match. This proves the claim. $\square$

This claim allows us to complete the induction by simply deleting P_n.