

# USAMO 2020/1

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TWITCH SOLVES ISL

Episode 16

## Problem

Let  $ABC$  be a fixed acute triangle inscribed in a circle  $\omega$  with center  $O$ . A variable point  $X$  is chosen on minor arc  $AB$  of  $\omega$ , and segments  $CX$  and  $AB$  meet at  $D$ . Denote by  $O_1$  and  $O_2$  the circumcenters of triangles  $ADX$  and  $BDX$ , respectively. Determine all points  $X$  for which the area of triangle  $OO_1O_2$  is minimized.

## Video

<https://youtu.be/r7j0oRtpErA>

## External Link

<https://aops.com/community/p15952768>

## Solution

We prove  $[OO_1O_2] \geq \frac{1}{4}[ABC]$ , with equality if and only if  $\overline{CX} \perp \overline{AB}$ .

**First approach (Bobby Shen).** We use two simultaneous inequalities:

- Let  $M$  and  $N$  be the midpoints of  $CX$  and  $DX$ . Then  $MN$  equals the length of the  $O$ -altitude of  $\triangle OO_1O_2$ , since  $\overline{O_1O_2}$  and  $\overline{DX}$  meet at  $N$  at a right angle. Moreover, we have

$$MN = \frac{1}{2}CD \geq \frac{1}{2}h_a$$

where  $h_a$  denotes the  $A$ -altitude.

- The projection of  $O_1O_2$  onto line  $AB$  has length exactly  $AB/2$ . Thus

$$O_1O_2 \geq \frac{1}{2}AB.$$

So, we find

$$[OO_1O_2] = \frac{1}{2} \cdot MN \cdot O_1O_2 \geq \frac{1}{8}h_a \cdot AB = \frac{1}{4}[ABC].$$

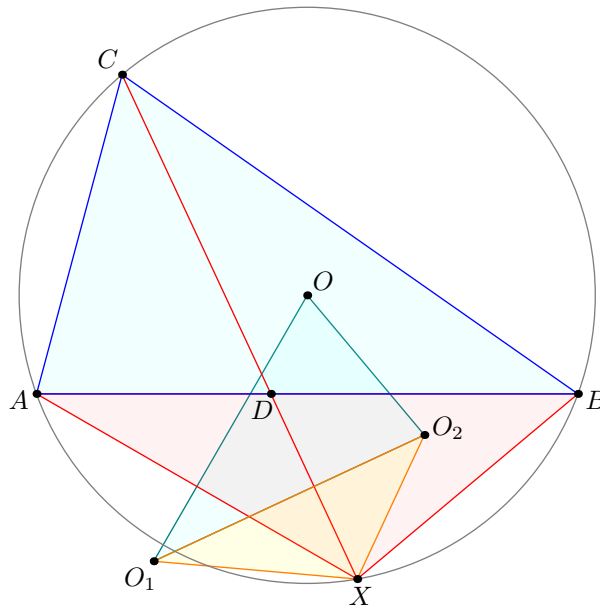
Note that equality occurs in both cases if and only if  $\overline{CX} \perp \overline{AB}$ . So the area is minimized exactly when this occurs.

**Second approach (Evan's solution).** We need two claims.

**Claim.** We have  $\triangle OO_1O_2 \sim \triangle CBA$ , with opposite orientation.

*Proof.* Notice that  $\overline{OO_1} \perp \overline{AX}$  and  $\overline{O_1O_2} \perp \overline{CX}$ , so  $\angle OO_1O_2 = \angle AXC = \angle ABC$ . Similarly  $\angle OO_2O_1 = \angle BAC$ .  $\square$

Therefore, the problem is equivalent to minimizing  $O_1O_2$ .



**Claim** (Salmon theorem). We have  $\triangle XO_1O_2 \sim \triangle XAB$ .

*Proof.* It follows from the fact that  $\triangle AO_1X \sim \triangle BO_2X$  (since  $\angle ADX = \angle XDB \implies \angle XO_1A = \angle XO_2B$ ) and that spiral similarities come in pairs.  $\square$

Let  $\theta = \angle ADX$ . The ratio of similarity in the previous claim is equal to  $\frac{XO_1}{XA} = \frac{1}{2\sin\theta}$ . In other words,

$$O_1O_2 = \frac{AB}{2\sin\theta}.$$

This is minimized when  $\theta = 90^\circ$ , in which case  $O_1O_2 = AB/2$  and  $[OO_1O_2] = \frac{1}{4}[ABC]$ . This completes the solution.