# USAMO 2020/1

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Twitch Solves ISL

Episode 16

## Problem

Let $ABC$ be a fixed acute triangle inscribed in a circle $\omega$ with center $O$. A variable point $X$ is chosen on minor arc $AB$ of $\omega$, and segments $CX$ and $AB$ meet at $D$. Denote by $O_1$ and $O_2$ the circumcenters of triangles $ADX$ and $BDX$, respectively. Determine all points $X$ for which the area of triangle $OO_1O_2$ is minimized.

## Video

https://youtu.be/r7j0oRtpErA

## Solution

We prove $[OO_1O_2] \geq \frac{1}{4}[ABC]$, with equality if and only if $\overline{CX} \perp \overline{AB}$.

**First approach (Bobby Shen)** We use two simultaneous inequalities:

- Let $M$ and $N$ be the midpoints of $CX$ and $DX$. Then $MN$ equals the length of the $O$-altitude of $\triangle OO_1O_2$, since $\overline{O_1O_2}$ and $\overline{DX}$ meet at $N$ at a right angle. Moreover, we have
$$MN = \frac{1}{2}CD \geq \frac{1}{2}h_a$$
where $h_a$ denotes the $A$-altitude.
- The projection of $O_1O_2$ onto line $AB$ has length exactly $AB/2$. Thus
$$O_1O_2 \geq \frac{1}{2}AB.$$

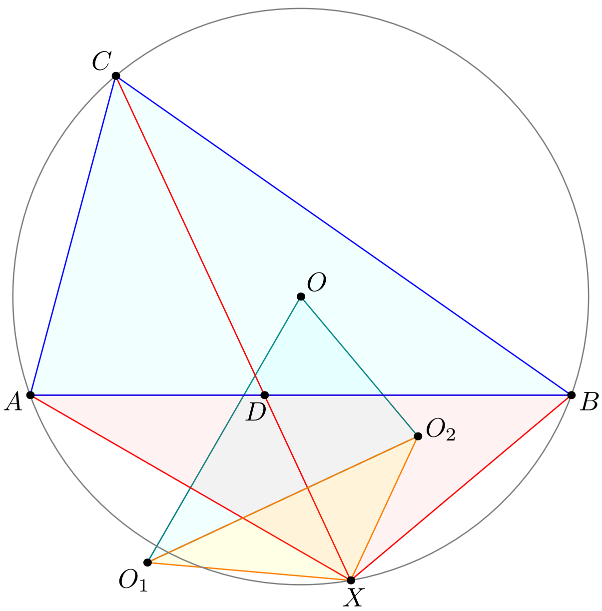
So, we find
$$[OO_1O_2] = \frac{1}{2} \cdot MN \cdot O_1O_2 \geq \frac{1}{8}h_a \cdot AB = \frac{1}{4}[ABC].$$
Note that equality occurs in both cases if and only if $\overline{CX} \perp \overline{AB}$. So the area is minimized exactly when this occurs.

**Second approach (Evan's solution)** We need two claims.

**Claim.** We have $\triangle OO_1O_2 \sim \triangle CBA$, with opposite orientation.

*Proof.* Notice that $\overline{OO_1} \perp \overline{AX}$ and $\overline{O_1O_2} \perp \overline{CX}$, so $\measuredangle OO_1O_2 = \measuredangle AXC = \measuredangle ABC$. Similarly $\measuredangle OO_2O_1 = \measuredangle BAC$. $\square$

Therefore, the problem is equivalent to minimizing $O_1O_2$.




**Claim** (Salmon theorem)**.** We have $\triangle XO_1O_2 \sim \triangle XAB$.

*Proof.* It follows from the fact that $\triangle AO_1X \sim \triangle BO_2X$ (since $\measuredangle ADX = \measuredangle XDB \implies \measuredangle XO_1A = \measuredangle XO_2B$) and that spiral similarities come in pairs. $\square$

Let $\theta = \angle ADX$. The ratio of similarity in the previous claim is equal to $\frac{XO_1}{XA} = \frac{1}{2\sin\theta}$. In other words,

$$O_1O_2 = \frac{AB}{2\sin\theta}.$$

This is minimized when $\theta = 90^\circ$, in which case $O_1O_2 = AB/2$ and $[OO_1O_2] = \frac{1}{4}[ABC]$. This completes the solution.